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Foundations of Quantum Physics: A General Realistic and Operational Approach

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We present a general formalism with the aim of describing the situation of an entity, how it is, how it reacts to experiments, how we can make statistics with it, and how it 'changes' under the influence of the rest of the universe. Therefore we base our formalism on the following basic notions: (1) the *states* of the entity, which describe the modes of being of the entity, (2) the *experiments* that can be performed on the entity, which describe how we act upon and collect knowledge about the entity, (3) the *outcomes* of our experiments, which describe how the entity and the experiments "are" and "behave" together, (4) the *probabilities*, which describe our repeated experiments and the statistics of these repeated experiments, and (5) the *symmetries*, which describe the interactions of the entity with the external world without being experimented upon. Starting from these basic notions we formulate the necessary derived notions: mixed states, mixed experiments and events, an eigenclosure structure describing the properties of the entity, an orthoclosure structure introducing an orthocomplementation, outcome determination, experiment determination, state determination, and atomicity giving rise to some of the topological separation axioms for the closures. We define the notion of subentity in a general way and identify the morphisms of our structure. We study specific examples in detail in the light of this formalism: a classical deterministic entity and a quantum entity described by the standard quantum mechanical formalism. We present a possible solution to the problem of the description of subentities within the standard quantum mechanical procedure using the tensor product of the Hilbert spaces, by introducing a new completed quantum mechanics in Hilbert space, were new 'pure' states are introduced, not represented by rays of the Hilbert space.

1. INTRODUCTION

Several scientists have worked in the past on the elaboration of axiomatic approaches to quantum mechanics and it would lead us too far to present in

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this paper an overview of all these approaches. It is possible, however, to indicate two specific lines that have inspired most of the 'traditional' of these approaches.

1.1. An Axiomatics for Standard Quantum Mechanics

The first line of inspiration was the recovery of standard quantum mechanics in an axiomatic way. In the standard quantum formalism a state $p_{\bar{c}}$ of an entity S is represented by the one-dimensional subspace or the ray $\bar{c}$ of a separable complex Hilbert space $\mathcal{H}$. An experiment e_H testing an observable is represented by a self-adjoint operator H on $\mathcal{H}$, and the set of outcomes of this experiment e_H is the spectrum $spec(H) \subset \mathbb{R}$. Measurable subsets $A \subset spec(H)$ represent the events (in the sense of probability theory) of outcomes. The interaction of the experiment e_H with the physical entity being in state $p_{\bar{c}}$ is described in the following way: (1) the probability for a specific event $A \subset spec(H)$ to occur if the entity is in a specific state $p_{\bar{c}}$ is given by $\langle c, P_A(c) \rangle$, where P_A is the spectral projection corresponding to A, c is the unit vector in state $\bar{c}$ and $\langle \cdot, \cdot \rangle$ is the inproduct in the Hilbert space $\mathcal{H}$; (2) if the outcome is contained in A, the state $p_{\bar{c}}$ is changed to $p_{\bar{d}}$, where $\bar{d}$ is the ray generated by $P_A(c)$.

This standard quantum mechanical formalism was the inspiration for most axiomatic approaches. In it, however, the structure of the set of states and of the experiments is derived from the structure of a complex separable Hilbert space. The presence of this Hilbert space is ad hoc, in the sense that there are no physically obvious and plausible reasons why the Hilbert space structure should be at the origin of both the structure of the state space as well as the structure of the experiments. This initiated the search for an axiomatic theory for quantum mechanics where the Hilbert space structure would be derived from more general and physically more plausible axioms (Birkhoff and Von Neumann, 1936; Zierler, 1961; Mackey, 1963; Piron, 1964; Jauch, 1968; Varadarajan, 1968; Beltrametti and Cassinelli, 1981). Due to the original focus (Birkhoff and Von Neumann, 1936) on the collection of 'experimental propositions' of a physical entity—with the conviction that such an 'experimental proposition' would be a good basic concept—most later axiomatics were constructed taking as their basic concept the set $\mathcal{L}$ of experimental propositions concerning an entity S. The first real breakthrough (Piron, 1964) came with a theorem of Constantin Piron, who proved that if $\mathcal{L}$ is a complete [Axiom 1], orthocomplemented [Axiom 2], atomic [Axiom 3] lattice, which is weakly modular [Axiom 4] and satisfies the covering law [Axiom 5], then each irreducible component of the lattice $\mathcal{L}$ can be represented as the lattice of all 'biorthogonal' subspaces of a vector space V over a division ring K (with some other properties satisfied that we shall not make explicit here). Such a vector space is called an 'orthomodular space' and also sometimes a 'generalized Hilbert space.' It can be shown that an infinite-dimensional orthomodular space over a division ring which is the real or complex numbers, or the quaternions, is a Hilbert space. For a long time there did not even exist any other example of an infinite-dimensional orthomodular space. The search for a further characterization of the real, complex, or quaternionic Hilbert space started (Wilbur, 1977). Then Keller constructed a nonclassical orthomodular space (Keller, 1980), and recently Solèr proved that any orthomodular space that contains an infinite orthonormal sequence is a real, complex, or quaternionic Hilbert space (Solèr, 1995; Holland, 1995). It is under investigation in which way this result of Solèr can be used to formulate new physically plausible axioms (Pulmannova, 1996; Holland, 1995; Aerts and Van Steirteghem, 1999).

1.2. An Operational Axiomatic Approach

A second line of inspiration could be called 'operationality.' Going along with the search for 'good' axioms was also the idea of founding the basic notions for this axiomatics in a physically clear and operational way. 'Operationality' means that the axioms should be introduced in such a way that they can be related to 'real physical operations' that can be performed in the laboratory. We have to say some words about this philosophical preoccupation with operationality. A first triumph for the 'operational method' was certainly the well-known analysis of the concept of simultaneity in physics by Albert Einstein that was also at the origin of the Einsteinian interpretation of relativity theory. Standard quantum mechanics is an example of a very nonoperational theory: the basic concept, the wave function, is in principle a mathematical object with no clear physical interpretation. The three approaches that have tried to formulate quantum mechanics operationally are the Geneva–Brussels approach (Jauch, 1968; Piron, 1964, 1976, 1989, 1990; Aerts, 1981, 1982, 1983a, b), the Amherst approach (Foulis and Randall, 1981; Foulis *et al.*, 1983, Randall and Foulis, 1976, 1978, 1981, 1983), and the Marburg approach (Ludwig, 1983, 1985). In all three approaches different concepts have been used as basic notions and different aspects of the possibility of an operational foundation have been investigated. The approach that we present in this paper has learned from these three and puts forward a new scheme that takes into account important results of the earlier approaches, but also gives new insights that have meanwhile grown out of the theoretical and experimental progress of the last decades (e.g., nonlocality is an experimental fact now and not a theoretical hypothesis any longer).

We also want to be explicitly critical of a general attitude that we would classify as 'naive operationalism.' As 'naive realism' believes that reality is just like it appears to us and in this way ignores the problem related to the way we gather knowledge about this reality, 'naive operationalism' believes that it is only our laboratory experiments that are 'real' and the rest is a construction out of the data and structure that we gather from these laboratory experiments. The extreme weight that naive operationalism puts on the laboratory experiments as the only candidates for foundational concepts is somewhat similar to the positivist and empiricist attitude in philosophy. It is known that to make experiments we need a theory and that as a consequence there is no nice hierarchy in the way naive operationalism proposes. We agree with the naive operationalists that our contact with reality is our experience and hence our experiments. In this sense it is good to make the effort and try to introduce as many possible basic concepts that are directly linked to these experiences and/or experiments. On the other hand we are convinced of the fact that the overall structure of reality, although it comes to us partially and in a fragmented way through our immediate experience with it, is revealed to us much more by the combination of a great many different experiences and by the way these different experiences form coherent wholes and are interrelated and also by the way they structure our long-term interaction with reality. In this sense we are also convinced of the fact that this overall contact with reality—of which our immediate sense experience and hence also our concrete laboratory experiments are only one aspect—reveals to us the global ontological structure of reality: 'the way things are' and 'what is the calculus of being.' It is by taking explicitly this fact into account that we will construct our foundational approach and in this sense we do not want to call it an 'operational' approach—because operationalism is often interpreted as what we have called naive operationalism—but a realistic and operational approach.

There is another aspect of our approach that we have to point out. As we have mentioned briefly in Section 1.1, most quantum axiomatics have been influenced by the original article of Birkhoff and Von Neumann, and as a consequence have chosen the concept of 'operational proposition' as their basic concept (called 'property' in the Geneva–Brussels approach). In the Amherst approach the concept of 'operation' is primary, but here one also tries to derive 'operational propositions' from this concept. We think that it is more fruitful to have more basic concepts than just the one of 'experimental proposition' or 'operation.' Therefore we will found our approach on five basic concepts and/or structures: states, experiments, outcomes, probabilities, and symmetries. These basic concepts express the naive operationalist foundational aspects, the laboratory experiments, but are also used to derive a 'calculus of being,' structuring the global reality as it is revealed to us from the overall structure of our experiences with it.

1.3. A Possible Solution of the Problem of the Description Subentities in Standard Quantum Mechanics

In standard quantum mechanics a subentity of a big entity is described within the tensor product procedure of the corresponding Hilbert spaces. As a consequence of the tensor product procedure there exists pure states (the so called nonproduct states) of the big entity that are such that if the big entity is in one of these pure states, the subentity is not in a pure state. This is a deep problem in standard quantum mechanics that has not been solved in a satisfactory way. In this paper we present a possible solution to this problem that comes to the definition of a new 'completed' quantum mechanics in Hilbert space, where new 'pure' states are introduced that cannot be represented by rays of the corresponding Hilbert spaces. We show how this solution follows naturally from the general approach that we have introduced and how it also is linked with earlier findings. We also want to mention that for the reader who is only interested in this newly introduced version of a 'completed' quantum mechanics, but does not want to study the new formalism in detail, that we have written Section 16 in a self-contained way. Such a reader might immediately proceed to Section 16.

The object of our description is the situation of a physical entity S in its most general way. The archetypical notions that we consider are the following:

The states: The physical entity S 'is' at each moment in a certain state p. In our approach the states describe the reality of the entity and the structure of the set of states expresses the main part of the 'calculus of being.'

The experiments: We gather knowledge about the entity by means of experiments $e, f, g, \ldots$ that we can perform on it. The structure of the set of these experiments expresses the main part of the way we investigate the reality of the entity.

The outcomes: The structure of the possible outcomes, i.e., the ways that the entity and the experiments performed on it can 'be' and 'behave' together, is at the root of our formalism.

The probabilities: For many entities these possibilities for certain outcomes can be structured in a probabilistic theory, probability being the representation of the relative frequencies of repeated experiments.

The symmetries: The entity changes also when we do not disturb it by a measurement and these changes are governed by symmetry principles on the reality of the entity, expressing its relation with the rest of the world.

These are the basic notions that we want to formalize in our approach. Derived concepts will be introduced step by step.

As we will see, an entity will be determined by a well-defined set of relevant states, a well-defined set of relevant experiments, a well-defined set of relevant outcomes, and the way in which these experiments interact with the entity in a state to give rise to an outcome. This entity corresponds to a physical phenomenon of the real physical world. In this way it is clear that what we often will classify, in our intuitive classification of phenomena of the real world, as the same phenomenon may correspond to different entities. Similarly, one entity may also correspond to different phenomena. In the traditional philosophical scheme it could be said that entities are 'models' of the phenomenon. However, we do not want to fix this traditional interpretation *a priori*, since we believe that a rigorous approach where an entity is defined by well-defined sets of the basic ontological notions of phenomena (states, experiments, outcomes, probabilities, and symmetries) may well lead, also philosophically, to a better 'ontological' classification.

2. BASIC NOTIONS

At a certain moment an entity S is in a certain state p. This state represents the reality of the entity at that moment. In this way we connect a well-defined set of states Σ to the entity S.

Basic Notion 1: States. Let S be an entity; then Σ is the set of states of this entity S. At each moment the entity S 'is' in a state $p \in \Sigma$, which will be referred to as the entity's 'actual' state. This state p represents the reality of the entity S at that moment. We shall denote states by symbols p, q, r, $\ldots$.

We gather our knowledge about the entity S and we act upon the entity by means of experiments that can be performed on S. A well-defined set of relevant experiments that are connected to a given entity S is denoted by $\mathscr{E}$ and we will denote experiments by $e, f, g, \ldots$.

Basic Notion 2: Experiments. Let S be an entity with a set of states Σ. The set of experiments that we use to gather knowledge about S and to act on S is denoted by $\mathscr{E}$. If an entity is in a certain state $p \in \Sigma$ and we perform an experiment $e \in \mathscr{E}$, then an outcome $x(e, p)$ occurs.

Different outcomes can possibly occur for an experiment e on an entity S in state p. The set of possible outcomes for e if S is in p is characteristic of the way in which the experiment and the entity interact, and will play a major role in our formalism. We denote this set of possible outcomes by $O(e, p)$.

Basic Notion 3: Outcomes. We denote by the nonempty set $O(e, p)$ the set of possible outcomes for experiment e if S is in the state p. We denote the set of all nonempty sets of possible outcomes for S being in state $p \in$

Σ and performing the experiment $e \; \varepsilon \; \mathscr{E}$ by $\mathbb{O} = \{O(e, p) | e \in \mathscr{E}, p \in \Sigma\}$. The set of possible outcomes of the experiment e will we denoted by $O(e) = \cup_{p \in \Sigma} O\ (e, p)$. The set of possible outcomes for all experiments on the entity S being in state p will be denoted by $O(p) = \cup_{e \in \mathscr{E}} O(e, p)$, and the set of all possible outcomes is denoted by $X = \cup_{p \in \Sigma,\ e \in \mathscr{E}} O(e, p)$.

In principle we could consider situations where $O(e, p) = \emptyset$, but in certain sense this would mean that the experiment e in question is not really applicable to the entity in this state p. Since this is a nonphysical situation, we make the hypothesis that for $p \in \Sigma$, $e \in \mathscr{E}$ we have $O(e, p) \neq \emptyset$.

We represent mathematically the entity S by a set of experiments $\mathscr{E}$, a set of states Σ, a set of outcomes X, and a nonempty set of outcomes $\mathbb{O} = \{O(e, p) | e \in \mathscr{E}, p \in \Sigma\}$. We denote the entity S by $S(\mathscr{E}, \Sigma, X, \mathbb{O})$ and will call it an 'experiment state outcome entity,' to indicate that the basic notions that we use to describe the entity are the experiments, the states, and the outcomes. Since we do not want to repeat each time the characterization 'experiment state outcome' we will just write 'the entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$' in those cases that it does not lead to confusion.

3. PREORDER AND ORTHOGONALITY

The archetypical situation that we consider is that of an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$ that 'is' in a state $p \in \Sigma$ and on which an experiment $e \in \mathscr{E}$ can be performed that gives rise an outcome $x(e, p) \in O(e, p)$. There are natural structures on $\mathscr{E} \times \Sigma$, on Σ, on $\mathscr{E}$, and on X. Our method to formalize these structures is the following: first we introduce the physical ideas and then we define the mathematical structure expressing these physical ideas. We do this in such a way that the mathematical structure is independent of the physical interpretation, but that, if interpreted, it gives rise to the original physical ideas.

Consider an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$ and two states $p, q \in \Sigma$. If it is such that for all experiments $e \in \mathscr{E}$ whenever S is in state p, the set of outcomes that can occur for an experiment e is contained in the set of outcomes that can occur for the experiment e if S is in the state q, we say that p 'implies' q and denote $p < q$. We call this implication the 'state implication.' This is the first example of a physical idea that we want to formalize. Let us first introduce a mathematical definition.

Definition 1 (preorder, equivalence). Consider a set Z and $a, b, c \in Z$. The relation $<$ is a preorder relation iff

$$a < a$$

$$a < b, b < c \Rightarrow a < c \qquad (1)$$

We say that two elements $a, b \in Z$ are equivalent, and we denote $a \approx b$, iff $a < b$ and $b < a$.

Definition 2 (state implication). For an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$, for $e \in \mathscr{E}$ and $p, q \in \Sigma$, we define

$$p <_e q \Leftrightarrow O(e, p) \subset O(e, q) \tag{2}$$

$$p < q \Leftrightarrow \forall f \in \mathscr{E}, \quad p <_f q \tag{3}$$

and we say respectively that p 'e-implies' q and that p 'implies' q, and call $<_e$ the 'e-state implication' and $<$ the state implication.

Theorem 1. For an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$, the state implications $<_e$ and $<$ introduced on Σ in Definition 2 are preorder relations.

Proof. Clearly for $p \in \Sigma$ we have $p < p$. Consider $p, q, r \in \Sigma$ such that $p < q$ and $q < r$. Then $\forall e \in \mathscr{E}$ we have $O(e, p) \subset O(e, q)$ and $O(e, q) \subset O(e, r)$. From this it follows that $\forall e \in \mathscr{E}$ we have $O(e, p) \subset O(e, q)$, which shows that $p < r$.

In a similar way we introduce natural implications on $\mathscr{E} \times \Sigma$ and on $\mathscr{E}$ that we call the 'central implication' and the 'experiment implication.'

Definition 3 (central implication, experiment implication). For an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$, for $(e, p), (f, q) \in \mathscr{E} \times \Sigma$, $e, f \in \mathscr{E}$, and $p \in \Sigma$ we define

$$(e, p) < (f, q) \Leftrightarrow O(e, p) \subset O(f, q) \tag{4}$$

$$e <_p f \Leftrightarrow O(e, p) \subset O(f, p) \tag{5}$$

$$e < f \Leftrightarrow \forall q \in \Sigma, \quad e <_q f \tag{6}$$

and we respectively say (e, p) 'implies' (f, q), e 'p-implies' f, and e 'implies' f, and call these implications respectively the 'central implication,' the 'p-experiment implication,' and the 'experiment implication.'

Theorem 2. For an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$, the implication relations $<$ and $<_p$ defined on $\mathscr{E} \times \Sigma$ and on $\mathscr{E}$ in Definition 3 are preorder relations.

Consider an entity $S(\mathscr{E},, \Sigma, X, \mathbb{O})$ and two states $p, q \in \Sigma$. If state p and state q can be 'distinguished' for the entity S, then we say that p and q are 'orthogonal' and we denote $p \perp q$. Before we formalize this physical concept of 'distinguished states' in our approach, let us introduce the mathematical concept of an orthogonality relation.

Definition 4 (orthogonality). Consider a set Z and $a, b \in Z$. The relation $\perp$ is an orthogonality relation iff

$$\begin{aligned} &a \not\perp a \\ &a \perp b \Rightarrow b \perp a \end{aligned} \tag{7}$$

Definition 5 (state orthogonality). For an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$ and for $p, q \in \Sigma$ we define

$$p \perp_e q \Leftrightarrow O(e, p) \cap O(e, q) = \emptyset \tag{8}$$

$$p \perp q \Leftrightarrow \exists e \in \mathscr{E}, \quad p \perp_e q \tag{9}$$

we say that p is 'e-orthogonal' to q if $p \perp_e q$, and p is 'orthogonal' to q if $p \perp q$. We call $\perp_e$ the 'e-state orthogonality' and $\perp$ the 'state orthogonality.'

Theorem 3. For an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$, the e-state orthogonality $\perp_e$ and the state orthogonality $\perp$ introduced on Σ in definition 5 is an orthogonality relation.

Proof. Clearly for $p \in \Sigma$ we have $p \not\perp_e p$ and $p \not\perp p$. Consider $p, q \in \Sigma$ such that $p \perp_e q$. Then $O(e, p) \cap O(e, q) = \emptyset$ and hence $q \perp_e p$. In an analogous way we show that $p \perp q$ implies $q \perp p$.

In a similar way we introduce natural orthogonality relations on $\mathscr{E} \times \Sigma$ and on $\mathscr{E}$ that we call the 'central orthogonality' and the 'experiment orthogonality.'

Definition 6 (central orthogonality, experiment orthogonality). For an entity S $(\mathscr{E}, \Sigma, X, \mathbb{O})$, for $(e, p), (f, q) \in \mathscr{E} \times \Sigma$ and $e, f \in \mathscr{E}$ we define

$$(e, p) \perp (f, q) \Leftrightarrow O(e, p) \cap O(f, q) = \emptyset \tag{10}$$

$$e \perp_p f \Leftrightarrow O(e, p) \cap O(f, p) = \emptyset \tag{11}$$

$$e \perp f \Leftrightarrow \exists p \in \Sigma, \quad e \perp_p f \tag{12}$$

We say that (e, p) is 'orthogonal' to (f, q), e is 'p-orthogonal' to f if $e \perp_p f$, and e is 'orthogonal' to f if $e \perp f$. We call the orthogonality relations respectively the 'central orthogonality,' the 'p-experiment orthogonality,' and the 'experiment orthogonality.'

There exists a natural orthogonality relation on the set of outcomes.

Definition 7 (outcome orthogonality). For an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$ and $x, y \in X$ we define

$$x \perp_{e,p} y \Leftrightarrow x, y \in O(e, p), \qquad x \neq y \tag{13}$$

$$x \perp y \Leftrightarrow \exists e \in \mathscr{E}, \quad p \in \Sigma, \quad x \perp_{e,p} y \tag{14}$$

we say that x is (e, p)-orthogonal to y if $x \perp_{e,p} y$ and x is orthogonal to y if $x \perp y$, and we call these relations respectively the '(e, p)-outcome orthogonality' and the 'outcome orthogonality.'

Theorem 4. Consider an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$. The central orthogonality, the p-experiment orthogonality, and the experiment orthogonality as introduced in Definition 6 and the outcome orthogonality as introduced in Definition 7 are orthogonality relations.

Proposition 1. For an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$ and $(e, p), (f, q) \in \mathscr{E} \times \Sigma$, $p, q \in \Sigma$, and $e, f \in \mathscr{E}$ we have

$$(e, p) < (f, q) \Rightarrow (e, p) \not\perp (f, q) \tag{15}$$

$$p < q \Rightarrow p \not\perp q \tag{16}$$

$$e < f \Rightarrow e \not\perp f \tag{17}$$

Moreover, the orthogonalities defined on $\mathscr{E} \times \Sigma$, $\mathscr{E}$, and Σ, have the following property:

$$a \perp b, \quad c < a, \quad d < b \Rightarrow c \perp d \tag{18}$$

We remark that a couple (e, p) is equivalent with a couple (f, q), and we denote $(e, p) \approx (f, q)$, iff $(e, p) < (f, q)$ and $(f, q) < (e, p)$, that two states $p, q \in \Sigma$ are equivalent, and we denote $p \approx q$, iff $p < q$, and $q < p$, and that two experiments $e, f \in \mathscr{E}$ are equivalent, and we denote $e \approx f$, iff $e < f$ and $f < e$.

Definition 8 (eigenstate, eigencouple). Suppose that we have an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$. We say that a state $p \in \Sigma$ is an 'eigenstate' for the experiment $e \in \mathscr{E}$ with 'eigenoutcome' $x(e, p)$ iff $O(e, p)$ is a singleton, and hence $O(e, p) = \{x(e, p)\}$. We also say in this case that (e, p) is an eigencouple with eigenoutcome $x(e, p)$.

If the state $p \in \Sigma$ is an eigenstate of the experiment $e \in \mathscr{E}$ with eigenoutcome $x(e, p)$, this means that the experiment e has a 'determined' outcome for S being in state p.

4. MIXED STATES, MIXED EXPERIMENTS, AND EVENTS

Often we are in a position that we 'lack knowledge' about the state p in which the entity S 'is' or about the experiment e that will be performed on the entity, or about the outcome that will occur. We should include a description of this possible lack of knowledge in our formalism. Suppose that we have an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$. Consider nonempty subsets $P \subset \Sigma$, E, $\subset \mathscr{E}$ and $A \subset X$. If we know that the entity is in one of the states of P, but we do not know in which one exactly, we are in a situation of 'lack of knowledge' about the state of the entity, and we will indicate this situation by the mixed state $p(P)$. If we know that an experiment of E will be performed, but we do not know exactly which one, we will indicate this situation by the mixed experiment $e(E)$. If one of the outcomes of A occurs, but we do not know which one exactly, we shall say that the event $x(A)$ connected to A occurs.

At first sight we would think that to one subset $P \subset \Sigma$ can correspond different situations of 'lack of knowledge' and hence different mixed states. Similarly one subset $E \subset \mathscr{E}$ can give rise to different mixed experiments and one subset $A \subset X$ to different events. This is in fact true, but we will choose to distinguish these different situations of lack of knowledge by means of the probability structure that we shall introduce later. At this stage of the formalism, we mean by mixed state (mixed experiment, event) the specification of a situation of lack of knowledge where we do not know its nature. We lack the knowledge and also lack the knowledge about the nature of this lack of knowledge. This is again a unique situation and it allows us to introduce mixed states, mixed experiments, and events in the following way.

Definition 9 (mixed experiments, mixed states, and events). Consider an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$, and given nonempty subsets $E \subset \mathscr{E}$, $P \subset \Sigma$, and $A \subset X$. The mixed experiment $e(E)$ consists in performing one of the experiments $f \in E$. The entity is in a mixed state $p(P)$ iff it is in one of the states $q \in P$. An event $x(A)$ occurs iff one of the outcomes $y \in A$ occurs.

Obviously we can consider a state q as being the trivial mixed state on the singleton $\{q\}$ and an experiment f to be the mixed experiment on the singleton $\{f\}$ and an outcome y to be the event connected with the singleton $\{y\}$.

Proposition 2. Suppose that we have an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$. For $f \in \mathscr{E}$, $q \in \Sigma$, and $y \in X$ we have

$$q = p(\{q\}), \qquad f = e(\{f\}), \qquad y = x(\{y\}) \tag{19}$$

for the mixed state $p(P)$ and the mixed experiment $e(E)$ we have

$$O(e(E), p) = \bigcup_{e \in E} O(e, p,) \qquad O(e, p(P)) = \bigcup_{p \in P} O(e, p) \tag{20}$$

$$O(e(E), p(P)) = \bigcup_{e \in E, p \in P} O(e, p) \tag{21}$$

Definition 10 (mixed entity). An entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$ is a 'mixed entity' iff there is a well-defined set of mixed experiments, mixed states, and events associated to the entity. We denote the sets of mixed experiments, mixed states, and events by $M(\mathscr{E})$, $M(\Sigma)$, and $M(X)$, respectively.

Definition 11. Suppose that we have a mixed entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$ with set of mixed states $M(\Sigma)$, and set of mixed experiments $M(\mathscr{E})$, and set of events $M(\mathrm{X})$. We generalize the preorder relations and the orthogonality relations that are defined on $\mathscr{E} \times \Sigma$, $\mathscr{E}$, Σ, and X to preorder relations and orthogonality relations defined on $M(\mathscr{E}) \times M(\Sigma)$, $M(\mathscr{E})$, $M(\Sigma)$, and $M(\mathrm{X})$. All the generalizations are straightforward, with the exception of the one for the events, which we will state explicitly: two events $x(A)$ and $x(B)$ are (e, p)-orthogonal iff $A \subset O(e, p)$ and $B \subset O(e, p)$ and $A \cap B = \emptyset$: we denote $x(A) \perp_{e,p} x(B)$. Two events $x(A)$ and $x(B)$ are orthogonal iff there exist $e \in \mathscr{E}$ and $p \in \Sigma$ such that $x(A) \perp_{e,p} x(B)$. We introduce a preorder relation on the set of events in a straightforward way: $x(A) < x(B) \Leftrightarrow A \subset B$.

We have to verify whether the preorder relation and the orthogonality relation, that we generalize on $M(\mathscr{E}) \times M(\Sigma)$, on $M(\mathscr{E})$, on $M(\Sigma)$, and on $M(X)$ coincide with the old preorder relation and orthogonality relation on $\mathscr{E} \times \Sigma$, $\mathscr{E}$, Σ and X. Since we have $e(\{f\}) = f$ for all $f \in \mathscr{E}$ and $p(\{q\}) = q$ for all $q \in \Sigma$, this is easily checked for the preorder relation and orthogonality relation. For the relations on $M(\mathscr{E})$, $M(\Sigma)$, and $M(X)$ we have to be more careful.

Proposition 3. Suppose that we have a mixed entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$ with set of mixed states $M(\Sigma)$, set of mixed experiments $M(\mathscr{E})$, and set of events $M(X)$. For states $q, r \in \Sigma$, experiments $f, g \in \mathscr{E}$, and outcomes $y, z, \in X$, we have

$$\begin{array}{ll} p(\{q\}) < p(\{r\}) \Leftrightarrow q < r & p(\{q\}) \perp_{e(\{f\})} p(\{r\}) \Leftrightarrow q \perp_f r \\ p(\{q\}) \perp p(\{r\}) \Leftrightarrow q \perp r & e(\{f\}) < e(\{g\}) \Leftrightarrow f < g \\ e(\{f\}) \perp_{p(\{q\})} e(\{g\}) \Leftrightarrow f \perp_q g & e(\{f\}) \perp e(\{g\}) \Leftrightarrow f \perp g \\ x(\{y\}) \perp_{e,p} x(\{z\}) \Leftrightarrow y \perp_{e,p} z & x(\{y\}) \perp x(\{z\}) \Leftrightarrow y \perp z \end{array} \tag{22}$$

Proof. Let us prove some of the equalities. We have $p(\{q\}) < p(\{r\}) \Leftrightarrow \forall f \in \mathscr{E}$: $O(e(\{f\}), p(\{q\})) \subset O(e(\{f\}), p(\{q\})) \Leftrightarrow \forall f \in \mathscr{E}$: $O(f, q) \subset O(f, r) \Leftrightarrow q < r$. We have $p(\{q\}) \perp p(\{r\}) \Leftrightarrow \exists e(E) \in M(\mathscr{E})$ such that $O(e(E), q) \cap O(e(E), r) = \emptyset$. But this is equivalent to the fact that $O(e, q) \cap O(e, r) = \emptyset$, $\forall e \in E$, which shows that $q \perp r$.

Proposition 4. Suppose that we have a mixed entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$ with set of mixed states $M(\Sigma)$, set of mixed experiments $M(\mathscr{E})$, and set of events $M(X)$. For mixed states $p(P), p(Q)$, mixed experiments $e(E), e(F)$, and events $x(A), x(B)$ we have

$$E \subset F \Rightarrow e(E) < e(F), \qquad P \subset Q \Rightarrow p(P) < p(Q) \tag{23}$$

$$(e(E), p(P)) < (e(F), p(Q)) \Leftrightarrow (e, p) < (e(F), p(Q)) \quad \forall e \in E, \; p \in P$$

$$(e(E), p(P)) \perp (e(F), p(Q)) \Leftrightarrow (e, p) \perp (f, q)$$
$$\forall e \in E, \; f \in F, \; p \in P, \; q \in Q$$

$$p(P) < p(Q) \Leftrightarrow p < p(Q) \quad \forall p \in P$$

$$e(E) < e(F) \Leftrightarrow e < e(F) \quad \forall e \in E$$

$$p(P) \perp_{e(E)} p(Q) \Leftrightarrow p \perp_{e(E)} q \quad \forall p \in P, \; q \in Q \tag{24}$$

$$e(E) \perp_{p(P)} e(F) \Leftrightarrow e \perp_{p(P)} f \quad \forall e \in E, \; f \in F \tag{25}$$

$$x(A) \perp_{e,p} x(B) \Leftrightarrow x \perp_{e,p} y \quad \forall x \in A, \; y \in B$$

$$p(P) \perp p(Q) \Rightarrow p \perp q \quad \forall p \in P, \; q \in Q$$

$$e(E) \perp e(F) \Rightarrow e \perp f \quad \forall e \in E, \; f \in F$$

$$x(A) \perp x(B) \Rightarrow x \perp y \quad \forall x \in A, \; y \in B$$

Proof. We have $(e(E), p(P)) < (e(F), p(Q)) \Leftrightarrow O(e(E), p(P)) \subset O(e(F), p(Q)) \Leftrightarrow \cup_{e \in E, p \in P} O(e, p) \subset O(e(F), p(Q)) \Leftrightarrow O(e, p) \subset O(e(F), p(Q))\ \forall e \in E,\ p \in P$. We also have $(e(E), p(P)) \perp (e(F), p(Q)) \Leftrightarrow O(e(E), p(P)) \cap O(e(F), p(Q)) = \emptyset \Leftrightarrow O(e(E), p(P)) \subset O(e(F), p(Q))^C \Leftrightarrow \cup_{e \in E,\ p \in P} O(e, p) \subset \cap_{f \in F, q \in Q} O(f, q)^C \Leftrightarrow O(e, p) \subset O(f, q)^C\ \forall e \in E, p \in P, f \in F, q \in Q \Leftrightarrow (e, p) \perp (f, q)\ \forall e \in E, p \in P, f \in F, q \in Q$. The other implications are proved in an analogous way.

Definition 12 (supremum and infimum). Suppose that Z is a set with a preorder relation $<$. Consider a set $\{a_j, j \in J\}$ of elements of Z. We say that $\vee_{j \in J} a_j$ is a supremum and $\wedge_{j \in J} a_j$ is an infimum iff for $b \in Z$ we have

$$a_j < b \quad \forall j \in J \Leftrightarrow \bigvee_{j \in J} a_j < b \tag{26}$$

$$b < a_j \quad \forall j \in J \Leftrightarrow b < \bigwedge_{j \in J} a_j \tag{27}$$

Theorem 5. Suppose that we have a mixed entity $S(\mathcal{E}, \Sigma, X, \mathcal{O})$ with set of mixed states $M(\Sigma)$, set of mixed experiments $M(\mathcal{E})$, and set of events $M(X)$. The mixed experiment $e(E) \in M(\mathcal{E})$ is a supremum of the set of experiments E for the preorder relation on $M(\mathcal{E})$, the mixed state $p(P)$ is a supremum for the set of states P for the preorder relation on $M(\Sigma)$, and the event $x(A)$ is a supremum for the set of outcomes A for the preorder relation on $M(X)$.

Proof. We have that $f < e(E)$ for $f \in E$. Suppose now that $f < g$ for all $f \in E$. This means that $O(f, p) \subset O(g, p)$ for all $p \in M(\Sigma)$ and $f \in E$. But then $\cap_{f \in E} O(f, p) \subset O(g, p)$ for all $p \in M(\Sigma)$. Hence $O(e(E), p) \subset O(g, p)$ for all $p \in M(\Sigma)$.

Definition 13. Suppose that we have a mixed entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$ with set of mixed states $M(\Sigma)$, set of mixed experiments $M(\mathscr{E})$, and set of events $M(X)$. Because of the foregoing proposition we shall also denote $e(E) = \vee_{f \in E} f$, $p(P) = \vee_{q \in P} q$, and $x(A) = \vee_{y \in x(A)} y$.

We have to remark that although $e(E) = \vee_{f \in E} f$ is well defined, it is not necessarily a unique supremum of the set E. The same remark holds for P.

Suppose that we consider a set of mixed states $P \subset M(\Sigma)$ of an entity S. Then we can again consider the situation of 'lack of knowledge' where we know that the entity is in one of the mixed states $q \in P$, but we do not know in which one: let us denote this mixed state (of mixed states) by $p(P)$. This is again a mixed state, but at first sight it is a type of mixed state that we have not yet considered explicitly in our formalism, namely a mixed state of mixed states. If this were really a new type of mixed state, we would arrive in a regressum ad infinitum, and this would be a problem. Luckily this is not the case. The new type $p(P)$ of mixed state is of the type that we have already introduced. Indeed, suppose that we denote an element $q \in P$ by $p(Q_q)$, where $Q_q \subset \Sigma$ is the set of states on which q is a mixed state. To the state $p(P)$ of lack of knowledge about the set of mixed states $q \in P$ corresponds the state of lack of knowledge about the set $\cup_{q \in P} Q_q$, i.e., $\vee_{q \in P} p(Q_q)$. And since the mixed state $p(P)$ exists, $\vee_{q \in P} p(Q_q) \in M(\Sigma)$.

Proposition 5. Suppose that we have a mixed entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$ with set of mixed states $M(\Sigma)$, set of mixed experiments $M(\mathscr{E})$, and set of events $M(X)$. We have

$$M(M(\Sigma)) \subset M(\Sigma), \qquad M(M(\mathscr{E})) \subset M(\mathscr{E}), \qquad M(M(X) \subset M(X) \quad (28)$$

Definition 14. Suppose that we have a mixed entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$ with set of mixed states $M(\Sigma)$, set of mixed experiments $M(\mathscr{E})$, and set of events $M(X)$. We will say that the entity is 'full' of mixed states iff there exists a mixed state $p(P)$ connected to each subset $P \subset \Sigma$. We will say that the entity is 'full' of mixed experiments iff there exists a mixed experiment $e(E)$ connected to each subset $E \subset \mathscr{E}$. We will say that an entity is 'full' of events iff there exists an event $x(A)$ for each $A \subset X$.

Definition 15 (complete preorder set). Consider a set Z with a preorder relation $<$; then Z is a 'complete' preorder set iff for each subset of Z there exists a supremum and an infimum.

Theorem 6. Suppose that we have a mixed entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$ with set of mixed states $M(\Sigma)$, set of mixed experiments $M(\mathscr{E})$, and set of events $M(X)$. If the entity is full of mixed states, then the preorder relation on $M(\Sigma)$ gives rise to a complete preorder set $M(\Sigma)$. If the entity is full of mixed measurements, the preorder relation on $M(\mathscr{E})$ gives rise to a complete preorder set $M(\mathscr{E})$. If the entity is full of events, the preorder relation on $M(X)$ gives rise to a complete preorder set $M(X)$. More concretely, for $P_i \subset \Sigma$ and $P = \cup_i P_i$, for $E_j \subset \mathscr{E}$ and $E = \cup_j E_j$, and for $A_k \subset X$ and $A = \cup_k A_k$ we have

$$p(P) = \bigvee_i p(P_i), \qquad e(E) = \bigvee_j e(E_j), \qquad x(A) = \bigvee_k x(A_k) \tag{29}$$

5. PROBABILITY

So far we have always referred to 'possible outcomes.' For most of the entities studied in physics these possibilities will be structured in such a way that they give rise to probabilities as limits of relative frequencies of repeated experiments. Indeed, for an entity S in state p, for an experiment e and for an outcome x we introduce the probability that, if the entity is in state p, the experiment e gives the outcome x, denoted by $\mu(e, p, x)$, as the limit of the relative frequency of the occurrence of the outcome x.

Definition 16. Suppose that we have a mixed entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$ with set of mixed states $M(\Sigma)$, set of mixed experiments $M(\mathscr{E})$, and set of events $M(X)$. Consider a map μ:

$$\mu: \quad M(\mathscr{E}) \times M(\Sigma) \times M(X) \to [0, 1], \qquad (e, p, x) \mapsto \mu(e, p, x) \tag{30}$$

We say that μ is a generalized probability measure iff for $e_i \in M(\mathscr{E})$, $p_j \in M(\Sigma)$, and $x_k \in M(X)$, countable sets, such that $e_i \perp e_l$ for $i \neq l$, $p_j \perp p_m$ for $j \neq m$, and $x_k \perp x_n$ for $k \neq n$, and such that $\vee_i e_i$ is a mixed experiment, $\vee_j p_j$ is a mixed state, and $\vee_k x_k$ is an event, we have

$$\mu(\vee_i e_i, \vee_j p_j, \vee_k x_k) = \sum_{i,j,k} \mu(e_i, p_j, x_k) \tag{31}$$

we also have that $x(O(e, p))$ is an event and we have

$$\mu(e, p, x(O(e, p))) = 1 \tag{32}$$

We say that the entity is probabilistic iff the different states of lack of knowledge are described by different generalized probability measures μ that correspond to limits of relative frequencies of outcomes in these states of lack of knowledge. Hence the probability $\mu(e, p, x)$ is the probability that the event x occurs when the entity S is in state p and the experiment e is performed in the state of lack of knowledge described by μ. This motivates the following definition:

Definition 17. Suppose that we have a mixed entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$ with set of mixed states $M(\Sigma)$, set of mixed experiments $M(\mathscr{E})$, and set of events $M(X)$. The entity S is probabilistic iff it has an associated well-defined set $\mathcal{M}$ of generalized probability measures. We denote a probabilistic entity by $S(\mathscr{E}, \Sigma, X, \mathbb{O}, \mathcal{M})$.

From now on we will only distinguish between states and mixed states, experiments and mixed experiments, and outcomes and events when it is explicitly necessary. The results that are valid for a general entity are of course also valid for a mixed entity, considered as a special type of entity.

6. STATE PROPERTY ENTITIES

In this section we want to introduce the concept of 'property' of an entity. We give a new description that is inspired by the way that properties are introduced in the Geneva–Brussels approach (Piron 1976, 1989, 1990; Aerts 1981, 1982, 1983). The main differences are (i) we distinguish between properties and 'testable' properties, a difference that has not been made in the earlier approaches, and (ii) we consider a property and a state as different concepts, while in the earlier approaches a state was represented by the set of all actual properties.

Let us consider an entity S. We remark that in this section S is not necessarily an 'experiment state outcome entity.' A property a of S is an attribute of S. The property a can be 'actual,' which means that S is in a state such that it 'has' the property a '*in acto*,' or 'potential,' which means that S is in a state such that it does not have the property a, but can eventually acquire it. Let us denote the set of properties corresponding to the entity S by $\mathscr{L}$. If the entity S is in a state p, we can consider the set $\xi(p)$ of all properties that are actual. We call $\xi(p)$ the property state connected to p. Let us call $\mathscr{T}$ the set of property states.

If for the entity being in an arbitrary state $p \in \Sigma$ we have that if $a \in \mathscr{L}$ is 'actual' then also $b \in \mathscr{L}$ is 'actual,' we say that a 'implies' b (or a is 'stronger than' b). This 'implication' introduces a 'preorder' relation on the set of properties $\mathscr{L}$. There exists also a natural preorder relation on the set of states for a state property entity. Indeed, if for two states $p, q \in \Sigma$, the set of properties $\xi(p)$ that is actual if the entity is in state p contains the set of properties $\xi(q)$ that is actual if the entity is in state q, then we say that p 'property implies' q.

We have now introduced all the necessary physical concepts to give a formal definition of an entity described by its states and properties.

Definition 18 (state property entity). We say that S is a state property entity iff it is characterized by a set of states Σ, a set of properties $\mathscr{L}$, and a function ξ:

$$\xi: \quad \Sigma \to \mathcal{P}(\mathcal{L}), \qquad p \mapsto \xi(p) \tag{33}$$

where $\xi(p)$ is the set of properties that are 'actual' if the entity S is in state p. We call ξ the state property function. Hence, for a property $a \in \mathcal{L}$ and a state $p \in \Sigma$ we have

$$a \text{ is actual if } S \text{ is in state } p \Leftrightarrow a \in \xi(p) \tag{34}$$

We call $\xi(p)$ the property state corresponding to p, and introduce $\mathcal{T} = \xi(\Sigma)$ the set of all property states. Further, we have that for $p, q \in \Sigma$ and $a, b \in \mathcal{L}$

$$p \prec q \Leftrightarrow \xi(q) \subset \xi(p) \tag{35}$$

$$a \prec b \Leftrightarrow \text{if for } p \in \Sigma \text{ we have } a \in \xi(p), \text{ then } b \in \xi(p) \tag{36}$$

and we say that p 'property implies' q and a 'implies' b and call this implication the 'property implication.' We denote a state property entity S by $S(\Sigma, \mathcal{L}, \xi)$.

Theorem 7. Consider a state property entity $S(\Sigma, \mathcal{L}, \xi)$. The implications on Σ and on $\mathcal{L}$ that are introduced in definition 18 are preorder relations.

Definition 19 (preorder set with an ordering set). Consider a set Z with a preorder relation $<$ and consider a set $U \subset \mathcal{P}(Z)$. We say that U is an ordering set for Z iff for $a, b \in Z$ we have $a < b$ iff whenever $u \in U$ such that $a \in u$ we have $b \in u$.

Theorem 8. Consider a state property entity $S(\Sigma, \mathcal{L}, \xi)$; then the set of property states $\xi(\Sigma) = \mathcal{T}$ is an ordering set for $\mathcal{L}, <$.

Proof. Consider $a, b \in \mathcal{L}$ such that $a \prec b$. Consider $p \in \Sigma$ such that $a \in \xi(p)$; then $b \in \xi(p)$. On the other hand suppose that for $\xi(p) \in \mathcal{T}$ we have $a \in \xi(p)$ implies $b \in \xi(p)$; then $a \prec b$.

It makes sense to identify equivalent properties. Indeed, equivalent properties are always 'actual' and potential together which makes it possible to indicate them as 'the same property' for the entity S. This is the reason that we introduce the following type of entity where such an identification has been made.

Definition 20 (identified state property entity). Consider a state property entity $S(\Sigma, \mathcal{L}, \xi)$. We say that $S(\Sigma, \mathcal{L}, \xi)$ is an 'identified' state property entity iff for $a, b \in \mathcal{L}$ we have $a \approx b \Rightarrow a = b$.

Theorem 9. For an identified state property entity $S(\Sigma, \mathcal{L}, \xi)$, the preorder relation on the set of properties is a partial order relation.

We have formalized the concept of state property entity. This is an entity for which we only consider the 'ontological' notions of 'state' and 'property' and how they are related. Properties can often also be directly tested. We will analyze now how this can be formalized. Consider an experiment e and a subset $A \subset O(e)$ of the outcome set of e. Suppose that we have a situation such that we are 'certain' that if we would perform e we find an outcome contained in A. Then it is possible to make correspond a 'property' $a(e, A)$ to this situation, $a(e, A)$ being 'actual' iff this situation is present. The property $a(e, A)$ that we have defined in this way is a 'testable' property.

Definition 21. Consider an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$ and an experiment $e \in \mathscr{E}$. For a set of outcomes $A \subset O(e)$ we introduce an e-testable property $a(A)$ such that

$$a(A) \text{ is actual if } S \text{ is in state } p \Leftrightarrow O(e, p) \subset A \tag{37}$$

We denote the set of e-testable properties of S by $\mathscr{L}(e)$.

We will see now that a state property entity for which the set of properties is $\mathscr{L}(e)$ for a given experiment e has more structure than a general state property entity. Let us investigate this additional structure. Although we need only one experiment to define a state property entity for which the set of properties is $\mathscr{L}(e)$, it will be more interesting—and we will not lose generality if we do—to investigate the structure of these entities for the case of an experiment state outcome entity. Let us first introduce a mathematical definition.

Proposition 6. Consider an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$ and for $e \in \mathscr{E}$ the set of e-testable properties $\mathscr{L}(e)$. Consider also the state property entity $S(\Sigma, \mathscr{L}(e), \xi_e)$. For $p \in \Sigma$, A, $B \subset O(e)$, $(A_i)_i$, $A_i \subset O(e)$, and $q, r \in \Sigma$ we have

$$a(A) \in \xi_e(p) \Leftrightarrow O(e, p) \subset A \tag{38}$$

$$a(A) \prec a(B) \Leftrightarrow \forall p \in \Sigma: \quad O(e, p) \subset A, \text{ then } O(e, p) \subset B \tag{39}$$

$$a(A_j) \in \xi_e(p) \forall j \Leftrightarrow a(\cap_i A_i) \in \xi_e(p) \tag{40}$$

$$q \prec r \Leftrightarrow q <_e r \tag{41}$$

Proof. The proofs of (38) and (39) are immediate consequences of Definitions 18 and 21. Let us prove (40). We have $a(A_j) \in \xi_e(p)\ \forall j \Leftrightarrow O(e, p) \subset A_j\ \forall j \Leftrightarrow O(e, p) \subset \cap_i A_i \Leftrightarrow a(\cap_i A_i) \in \xi_e(p)$. Let us prove (41). Suppose that $q \prec r$. This means that $\xi_e(r) \subset \xi_e(q)$. We have that $a(O(e, r)) \in \xi_e(r)$ and hence $a(O(e, r)) \in \xi_e(q)$. From this, using (38), it follows that $O(e, q) \subset O(e, r)$ and as a consequence we have $q <_e r$. Suppose now that $q <_e r$ and hence $O(e, q) \subset O(e, r)$. Consider $a(A) \in \xi_e(r)$. Then we have $O(e, q) \subset O(e, r) \subset A$ and hence $a(A) \in \xi_e(q)$. This shows that $\xi_e(r) \subset \xi_e(q)$ and as a consequence $q \prec r$.

Theorem 10. Consider an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$ and for $e \in \mathscr{E}$ the set of e-testable properties $\mathscr{L}(e)$. The preorder set of properties of the state property entity $S(\Sigma, \mathscr{L}(e), \xi)$ is a complete preorder set (see Definition 15) with a maximal element $I = a(O(e))$ and minimal element $0 = a(\emptyset)$.

Proof. Consider $(a_i)_i$, $a_i \in \mathscr{L}(e)$. For each a_i there exists a set of outcomes $A_i \subset O(e)$ such that $a(A_i) = a_i$. Consider the e-testable property $a(\cap_i A_i)$. Let us show that $a(\cap_i A_i)$ is an infimum for the set $(a_i)_i$. From (40) it follows that $a(\cap_i A_i) \prec a(A_j)\ \forall j$. Suppose that $a(A) \prec a(A_j)\ \forall j$. Consider $O(e, p) \subset A$. From this it follows that $O(e, p) \subset A_j\ \forall j$ and hence $O(e, p) \subset \cap_i A_i$. This shows, taking into account (39), that $a(A) \prec a(\cap_i A_i)$. So $a(\cap_i A_i)$ is an infimum for the set $(a_i)_i$. There is a natural construction for a supremum that consists in taking the infimum of all elements that are 'implied' by all elements of the considered set. We remark, however, that this supremum depends in principle on all elements of the preordered set. Let us identify a maximal and a minimal element. We have $O(e, p) \subset O(e)$ always and hence $a(A) \prec a(O(e))$ for an arbitrary $A \subset O(e)$. This shows that $I = a(O(e))$ is a maximal element of $\mathscr{L}(e)$. On the contrary, $O(e, p) \subset \emptyset$ never, which shows that $a(\emptyset) \prec a(A)$ for an arbitrary $A \subset O(e)$. Hence $a(\emptyset)$ is a minimal element of $\mathscr{L}(e)$.

We will introduce now the mathematical concept of a 'state property system' and then show that the state property entity $S(\Sigma, \mathscr{L}(e), \xi_e)$ (once properties are identified) is well described by a state property system.

Definition 22 (state property system). We say that $(\Sigma, \prec, \mathscr{L}, \prec, \wedge, \vee, \xi)$, or more concisely $(\Sigma\ \mathscr{L}, \xi)$, is a state property system iff $(\Sigma, \prec)$ is a preordered set, $(\mathscr{L}, \prec, \wedge, \vee)$ is a complete lattice, and ξ is a function:

$$\xi: \quad \Sigma \to \mathscr{P}(\mathscr{L}), \qquad p \mapsto \xi(p) \tag{42}$$

For $p \in \Sigma$, I the maximal element and 0 the minimal element of $\mathscr{L}$, and $a_i \in \mathscr{L}$, we have

$$I \in \xi(p), \qquad 0 \notin \xi(p), \qquad a_i \in \xi(p) \Leftrightarrow \wedge_i a_i \in \xi(p) \tag{43}$$

Further, for $p, q \in \Sigma$ and for $a, b, a_i \in \mathscr{L}$, we have

$$p \prec q \Leftrightarrow \xi(q) \subset \xi(p) \tag{44}$$

$$a \prec b \Leftrightarrow a \in \xi(r), \quad \text{then } b \in \xi(r) \quad \forall r \in \Sigma \tag{45}$$

Theorem 11. Consider an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$ and for $e \in \mathscr{E}$ we consider the identified state property entity $S(\Sigma, \mathscr{L}(e), \xi_e)$; then $(\Sigma, \mathscr{L}(e), \xi_e)$ is a state property system.

Proof. We only have to remark that for an identified state property entity, the infimum and supremum that are constructed in (15) are the infimum and supremum for the partially ordered set $\mathcal{L}(e)$. This makes $\mathcal{L}(e)$ into a complete lattice with maximal element I and minimal element 0.

We will now show that the state property systems are naturally connected to closure structures on the set of states.

Definition 23 (Cartan map). Consider a state property entity $S(\Sigma,\mathcal{L},\xi)$. We introduce the function

$$\kappa: \quad \mathcal{L} \rightarrow \mathcal{P}(\Sigma), \quad a \mapsto \kappa(a) \tag{46}$$

$$p \in \kappa(a) \Leftrightarrow a \in \xi(p) \tag{47}$$

which we call the 'Cartan map' (Aerts, 1981, 1982, 1983; Piron, 1990).

The meaning of the Cartan map is the following: $\kappa(a)$ is the set of all states that make a actual. Let us now introduce the eigenmaps.

Definition 24 (eigenmaps on the states). Consider an experiment state outcome entity $S(\mathcal{E}, \Sigma, X, \mathbb{O})$. For $e \in \mathcal{E}$ and $A \subset O(e)$, we define a map eig_e, which we shall call the eigenmap corresponding to the experiment e:

$$eig_e: \quad \mathcal{P}(O(e)) \rightarrow \mathcal{P}(\Sigma), \qquad A \mapsto eig_e(A) \tag{48}$$

$$p \in eig_e(A) \Leftrightarrow O(e, p) \subset A \tag{49}$$

The eigenmap eig_e connects a subset A of outcomes of e with a subset of states $eig_e(A)$ such that if the entity S is in one of the states of $eig_e(A)$, one of the outcomes of A occurs with certainty.

Proposition 7. Consider an entity $S(\mathcal{E}, \Sigma, X, \mathbb{O})$ and for $e \in \mathcal{E}$ consider the state property entity $S(\Sigma, \mathcal{L}(e), \xi_e)$. For $a(A) \in \mathcal{L}(e)$ we have

$$\kappa(a) = eig_e(A) \tag{50}$$

Before we proceed, let us point out some of the properties of the Cartan map and of the eigenmaps.

Proposition 8. Consider a state property entity $S(\Sigma, \mathcal{L}, \xi)$. For $a \in \mathcal{L}$ we have

$$a \prec b \Leftrightarrow \kappa(a) \subset \kappa(b) \tag{51}$$

Proposition 9. Consider an entity $S(\mathcal{E}, \Sigma, X, \mathbb{O})$. The map eig_e introduced in Definition 24 satisfies the following properties:

$$eig_e(\emptyset) = \emptyset \tag{52}$$

$$eig_e(O(e)) = \Sigma \tag{53}$$

$$eig_e(\cap_i A_i) = \cap_i \, eig_e(A_i) \tag{54}$$

Proof. $p \in eig_e(\cap_i A_i) \Leftrightarrow O(e, p) \subset \cap_i A_i \Leftrightarrow O(e, p) \subset A_i \ \forall_i \Leftrightarrow p \in eig_e(A_i) \ \forall_i \Leftrightarrow p \in \cap_i eig_e(A_i)$.

Let us introduce some definitions.

Definition 25 (closure system). Consider a set W. We say that $\mathcal{F} \subset \mathcal{P}(W)$ is a closure system iff

$$\emptyset \in \mathcal{F} \tag{55}$$

$$W \in \mathcal{F} \tag{56}$$

$$F_i \in \mathcal{F} \Rightarrow \bigcap_i F_i \in \mathcal{F} \tag{57}$$

Definition 26 (closure operator). Consider a set W. We say that cl is a closure operator on W iff, for $K, L \subset W$, we have

$$K \subset cl(K) \tag{58}$$

$$K \subset L \Rightarrow cl(K) \subset cl(L) \tag{59}$$

$$cl(cl(K)) = cl(K) \tag{60}$$

$$cl(\emptyset) = \emptyset \tag{61}$$

Proposition 10. If a set W is equipped with a closure operator cl and we define a subset $F \subset W$ to be closed iff $cl(F) = F$, then the set $\mathcal{F}$ of closed subsets of W forms a closure system on W. Suppose on the other hand that we consider a closure system $\mathcal{F}$ on W. If, for an arbitrary $K \subset \Sigma$, we define

$$cl(K) = \bigcap_{\kappa \subset F, F \in \mathcal{F}} F \tag{62}$$

then cl is a closure operator on W, and $\mathcal{F}$ is the set of closed subsets of W defined by this closure operator.

Proof. First we prove (57). We have that $cl(\cap_i F_i) \subset cl(F_i) \ \forall_i$ implies $cl(\cap_i F_i) \subset \cap_i cl(F_i) = \cap_i F_i \subset cl(\cap_i F_i)$. Now we show that (15) defines a closure operator on W. So consider $K, L \subset W$. Clearly $cl(\emptyset) = \emptyset$, $K \subset cl(K)$ and if $K \subset L$ then $cl(K) \subset cl(L)$. If $F \in \mathcal{F}$, then $cl(F) = F$, whence $cl(K) \in \mathcal{F}$ implies $cl(cl(K)) = cl(K)$. This shows that cl is a closure operator. Consider a set K such that $cl(K) = K$; then $K = \cap_{K \subset F, F \in \mathcal{F}} F$, and hence $K \in \mathcal{F}$. It follows that $\mathcal{F}$ is the set of closed subsets for this cl.

Theorem 12. Consider an entity $S(\mathcal{E}, \Sigma, X, \mathcal{O})$, and eigenmaps eig_e, $e \in \mathcal{E}$. Let us denote the image of an eigen map eig_e by $\mathcal{F}(e)$; in other words,

$$\mathcal{F}(e) = \{F \subset \Sigma \mid \exists A \subset O(e), F = eig_e(A)\} \tag{63}$$

then $\mathcal{F}(e)$ is a closure system on Σ. We will call the elements of $\mathcal{F}(e)$ the e-eigenclosed sets.

Proof. We have $eig_e(\emptyset) = \emptyset$ and $eig_e(O(e)) = \Sigma$. Consider $F_i \in \mathcal{F}(e)$. Then $F_i = eig_e(A_i)$. We have $\cap_i F_i = \cap_i eig_e(A_i) = eig_e(\cap_i A_i)$. This shows that $\cap_i F_i \in \mathcal{F}(e)$.

Theorem 13. Consider an entity $S(\mathcal{E}, \Sigma, X, \mathbb{O})$ and for $e \in \mathcal{E}$ consider the state property entity $S(\Sigma, \mathcal{L}(e), \xi_e)$. We have

$$\kappa(\mathcal{L}(e) = \mathcal{F}(e) \tag{64}$$

Proof. Consider $F \in \kappa(\mathcal{L}(e))$. Then there exists $a(A) \in \mathcal{L}(e)$ such that $\kappa(a(A)) = F$. From (50) it follows that $\kappa(a(A)) = eig_e(A)$ and hence $F \in \mathcal{F}(e)$. Suppose now that $F \in \mathcal{F}(e)$; then we have $F = eig_e(A)$ for some $A \subset O(e)$. Again from (50) it follows that $F = \kappa(a(A))$ and hence $F \in \kappa(\mathcal{L}(e))$.

7. STATE PROPERTY SYSTEMS AND CLOSURE SPACES

From Theorem 11 it follows that an identified state property entity $S(\Sigma, \mathcal{L}(e), \xi_e)$ is represented mathematically by a state property system. From Theorems 12 and 13 it follows that there is a closure system on the states connected with the state property entity $S(\Sigma, \mathcal{L}(e), \xi_e)$. We will see now that the connection between state property systems and closure systems is even much more intimate than we would expect from the foregoing section. Since we will encounter the mathematical concepts of state property system and closure system again for the description of entities, we want to make the results of this section independent of the physical content. Therefore we introduce some concepts again that have been introduced earlier within a specific physical context.

Proposition 11. Suppose that $(\Sigma, \mathcal{L}, \xi)$ is a state property system. We introduce the function κ, the Cartan map:

$$\kappa: \quad \mathcal{L} \to \mathcal{P}(\Sigma), \qquad a \mapsto \kappa(a) = \{p \mid a \in \xi(p\} \tag{65}$$

For $a, b, a_i \in \mathcal{L}$ we have

$$\kappa(\mathrm{I}) = \Sigma \tag{66}$$

$$\kappa(0) = \emptyset \tag{67}$$

$$a \prec b \Leftrightarrow \kappa(a) \subset \kappa(b) \tag{68}$$

$$\kappa(\wedge_i a_i) = \bigcap_i \kappa(a_i) \tag{69}$$

Proof. Since $\mathrm{I} \in \xi(p)\ \forall p \in \Sigma$, we have $\kappa(\mathrm{I}) = \Sigma$. Since $0 \notin \xi(p)\ \forall p \in \Sigma$, we have $\kappa(0) = \emptyset$. Take $a \prec b$ and consider $p \in \kappa(a)$. Then $a \in \xi(p)$ and since $a \prec b$ we have $b \in \xi(p)$. This implies that $p \in \kappa(b)$. Hence we have shown that $\kappa(a) \subset \kappa(b)$. Take now $\kappa(a) \subset \kappa(b)$. Consider $p \in \Sigma$ such that $a \in \xi(p)$. Then $p \in \kappa(a)$ and hence $p \in \kappa(b)$. From this it follows that $b \in \xi(p)$. This means that $a \prec b$. We have $\wedge_i a_i \prec a_j \ \forall_j$. This implies that $\kappa(\wedge_i a_i) \subset \kappa(a_j) \ \forall_j$. Hence $\kappa(\wedge_i a_i) \subset \cap_i \kappa(a_i)$. Take now $p \in \cap_i \kappa(a_i)$; then $p \in \kappa(a_j) \ \forall_j$. Hence $a_j \in \xi(p) \ \forall_j$, which implies that $\wedge_i a_i \in \xi(p)$. From this it follows that $p \in \kappa(\wedge_i a_i)$. As a consequence we have $\cap_i \kappa(a_i) \subset \kappa(\wedge_i a_i)$. This shows that $\kappa(\wedge_i a_i) = \cap_i \kappa(a_i)$.

Theorem 14. Suppose that $(\Sigma, \mathcal{L}, \xi)$ is a state property system. Let us introduce $\mathcal{F} = \{\kappa(a) \mid a \in \mathcal{L}\}$. Then $\mathcal{F}$ is a closure system on Σ.

Proof. From the foregoing theorem it follows that $\Sigma \in \mathcal{F}$ and $\emptyset \in \mathcal{F}$. Consider $F_i \in \mathcal{F}$. Then there exists $a_i \in \mathcal{L}$ such that $\kappa(a_i) = F_i$. We have $\kappa(\wedge_i a_i) = \cap_i \kappa(a_i) = \cap_i F_i$. This shows that $\cap_i F_i \in \mathcal{F}$.

This theorem shows that to a state property system there corresponds in a natural way a closure system on the set of states, where the properties are represented by the closed subsets. We can show that to each closure system on the set of states there corresponds also a state property system.

Theorem 15. Consider a set Σ with a closure system $\mathcal{F}$ on Σ. We define $\mathcal{L}$ in the following way. The elements of $\mathcal{L}$ are the elements of $\mathcal{F}$; hence $\mathcal{L} = \mathcal{F}$, where we identify the maximal element I of $\mathcal{L}$ with Σ and the minimal element 0 of $\mathcal{L}$ with $\emptyset$. For $F, G \in \mathcal{L}$ we define $F \prec G$ iff $F \subset G$. For $F_i \in \mathcal{L}$ we define $\wedge_i F_i = \cap_i F_i$ and $\vee_i F_i = cl(\cup_i F_i)$. We introduce the function ξ in the following way:

$$\xi: \quad \Sigma \to \mathcal{P}(\mathcal{L}), \qquad p \mapsto \{F \mid F \in \mathcal{F}, p \in F\} \tag{70}$$

We introduce a preorder relation on Σ. For $p, q \in \Sigma$ we define

$$p \prec q \Leftrightarrow \xi(q) \subset \xi(p) \tag{71}$$

Then $(\Sigma, \mathcal{L}, \xi)$ is a state property system.

Proof. It is easy to show that $\mathcal{L}, \prec \wedge, \vee$ is a complete lattice. We have $I \in \xi(p) \ \forall p \in \Sigma$ and $0 \notin \xi(p) \ \forall p \in \Sigma$. Suppose that $F_i \in \xi(p) \ \forall i$. This means that $p \in F_i \ \forall i$ and hence $p \in \cap_i F_i$. As a consequence we have $\cap_i F_i \in \xi(p)$. Let us verify that $\mathcal{T} = \{\xi(p) \mid p \in \Sigma\}$ is an ordering set. Suppose that $F, G \in \mathcal{L}$ and $F \subset G$. Consider p such that $F \in \xi(p)$ and hence $p \in F$. This implies that $p \in G$ and hence $G \in \xi(p)$. Suppose now that for $p \in \Sigma$ we have that $F \in \xi(p)$ implies that $G \in \xi(p)$. Consider then $p \in F$ and hence $F \in \xi(p)$. Then $G \in \xi(p)$ and as a consequence $p \in G$. This shows that $F \subset G$. It is easy to verify that $\prec$ is a preorder on Σ.

Theorems 14 and 15 show that there is a natural correspondence between state property systems and closure systems. Let us introduce the morphisms of these structures. Consider two state property systems $(\Sigma, \mathcal{L}, \xi)$ and $(\Sigma', \mathcal{L}', \xi')$. As we have explained, both state property systems describe respectively identified state property entities $S(\Sigma, \mathcal{L}, \xi)$ and $S'(\Sigma', \mathcal{L}', \xi')$. We will arrive at the notion of morphism by analyzing the situation where the entity S is a subentity of the entity S'. If the entity S is a subentity of the entity S', then we have natural requirements that have to be satisfied.

(i) If the entity S' is in a state p', then also the entity S is in a state $m(p')$, and all states of S are of this type. This defines a surjective function m from the set of states of S' to the set of states of S.

(ii) If we consider a property a of the entity S, then there corresponds a property $n(a)$ of the entity S' with this property a. This defines a function n from the set of properties of S to the set of properties of S'.

Requirement of Covariance Connected to the Relation of 'Being a Subentity' of 'an Entity'

The most important and fundamental requirement as to the concept of subentity and the derived concept of morphism will be put forward now. It is a requirement of 'covariance' on the ontological level. We want to express now that the reality of the physical phenomenon described by the entity or by the subentity, depending of whether we consider a bigger piece (the entity) or smaller piece (the subentity) of this reality, is independent of this choice.

This implies that if the entity S' is in state p', then the subentity S is in state $m(p')$. Suppose that the property a is actual; then also the property $n(a)$ must be actual. This shows that we must have

$$a \in \xi(m(p')) \Leftrightarrow n(a) \in \xi'(p') \tag{72}$$

We are ready now to present a formal definition of a morphism.

Definition 27. Consider two state property systems $(\Sigma, \mathcal{L}, \xi)$ and $(\Sigma', \mathcal{L}', \xi')$. We say that a couple of functions (m, n) is a morphism iff m is a function

$$m: \quad \Sigma' \to \Sigma, \quad p' \mapsto m(p') \tag{73}$$

and n is a function

$$n: \quad \mathcal{L} \to \mathcal{L}', \quad a \mapsto n(a) \tag{74}$$

such that

$$a \in \xi(m(p')) \Leftrightarrow n(a) \in \xi'(p') \tag{75}$$

Proposition 12. Consider two state property systems $(\Sigma, \mathcal{L}, \xi)$ and $(\Sigma',$

$\mathscr{L}'$, ξ'). The couple of functions (m, n) as introduced in definition 27 is a morphism iff we have

$$\xi \circ m = n^{-1} \circ \xi' \tag{76}$$

Proposition 13. Consider two state property systems $(\Sigma, \mathscr{L}, \xi)$ and $(\Sigma', \mathscr{L}', \xi')$ connected by a morphism (m, n). For $p', q' \in \Sigma'$, $a, b \in \mathscr{L}$, $a_i \in \mathscr{L}$, we have

$$a \prec b \Rightarrow n(a) \prec n(b) \tag{77}$$

$$n(\wedge_i a_i) = \wedge_i n(a_i) \tag{78}$$

$$n(\mathrm{I}) = \mathrm{I}' \tag{79}$$

$$n(0) = 0' \tag{80}$$

$$p' \prec q' \Rightarrow m(p') \prec m(q') \tag{81}$$

Proof. Suppose that $a \prec b$. Consider $\xi(p')$ such that $n(a) \in \xi'(p')$. Then we have $a \in \xi(m(p'))$. Since $a \prec b$ we have $b \in \xi(m(p'))$. From this it follows that $n(b) \in \xi'(p')$. So we have shown that $n(a) \prec n(b)$. We have $\wedge_i a_i \prec a_j \ \forall j$ and hence $n(\wedge_i a_i) \prec n(a_j) \ \forall j$. This shows that $n(\wedge_i a_i) \prec \wedge_i n(a_i)$. We still have to show that $\wedge_i n(a_i) \prec n(\wedge_i a_i)$. Consider $\xi'(p')$ such that $\wedge_i n(a_i) \in \xi'(p')$. This implies that $n(a_j) \in \xi'(p') \ \forall j$. But from this it follows that $a_j \in \xi(m(p')) \ \forall j$ and hence $\wedge_i a_i \in \xi(m(p'))$. As a consequence we have $n(\wedge_i a_i) \in \xi'(p')$. But then we have shown that $\wedge_i n(a_i) \prec n(\wedge_i a_i)$. As a consequence we have $n(\wedge_i a_i) = \wedge_i n(a_i)$. We have $n(\mathrm{I}) \prec \mathrm{I}'$. Consider $p' \in \Sigma'$; then $\mathrm{I}' \in \xi'(p')$. We also have $\mathrm{I} \in \xi(m(p'))$, which implies that $n(\mathrm{I}) \in \xi'(p')$). This proves that $\mathrm{I}' \prec n(\mathrm{I})$ and hence $n(\mathrm{I}) = \mathrm{I}'$. In an analogous way we prove that $n(0) = 0'$. Suppose that $p' \prec q'$. We then have $\xi'(q') \subset \xi'(p')$. From this it follows that $n^{-1}(\xi'(q')) \subset n^{-1}(\xi'(p'))$. As a consequence we have $\xi(m(q')) \subset \xi(m(p'))$ and this implies that $m(p') \prec m(q')$.

Proposition 14. Suppose that we have two state property systems $(\Sigma, \mathscr{L}, \xi)$ and $(\Sigma', \mathscr{L}', \xi')$ connected by a morphism (m, n). Consider the Cartan maps κ and κ' that connect these state property systems with their corresponding closure systems $(\Sigma, \mathscr{F})$ and $(\Sigma', \mathscr{F}')$. For $a \in \mathscr{L}$ we have

$$m^{-1}(\kappa(a)) = \kappa'(n(a)) \tag{82}$$

Proof. We have $p' \in m^{-1}(\kappa(a)) \Leftrightarrow m(p') \in \kappa(a) \Leftrightarrow a \in \xi(m(p')) \Leftrightarrow n(a) \in \xi'(p') \Leftrightarrow p' \in \kappa'(n(a))$.

Theorem 16. Suppose that we have two state property systems $(\Sigma\ \mathscr{L}, \xi)$ and $(\Sigma', \mathscr{L}', \xi')$ connected by a morphism (m, n) and the Cartan maps κ and κ' that connect these state property systems with their corresponding closure systems $(\Sigma, \mathcal{F})$ and $(\Sigma', \mathcal{F}')$. The function m is a continuous function for the closure systems.

Proof. Take a closed subset $F \in \mathcal{F}$ and consider $m^{-1}(F)$. Since $F \in \mathcal{F}$ we have $a \in \mathcal{L}$ such that $\kappa(a) = F$. From the foregoing theorem we have $m^{-1}(F) = m^{-1}(\kappa(a)) = \kappa'(n(a)) \in \mathcal{F}'$. This shows that m is continuous.

We are now at the point of making explicit the powerful representation that the closure system gives for a state property system. Let us identify the morphisms of the closure systems that correspond to the morphisms that we have introduced in the state property systems.

Theorem 17. Suppose that we have two closure systems $(\Sigma, \mathcal{F})$ and $(\Sigma', \mathcal{F}')$ and a continuous function $m: \Sigma' \to \Sigma$. Consider the state property systems $(\Sigma, \mathcal{L}, \xi)$ and $(\Sigma', \mathcal{L}', \xi')$ corresponding to these two closure systems, as proposed in Theorem 15. If we define the couple (m, n) such that

$$n = m^{-1} \tag{83}$$

then (m, n) is a morphism between the two state property systems.

Proof. We have to prove that the couple (m, m^{-1}) satisfies that properties of a state property morphism as put forward in Definition 27. Since m is continuous we have that m^{-1} is a function from $\mathcal{F}$ to $\mathcal{F}'$. Let us show now formula (75) using the definition of ξ and ξ' as put forward in Theorem 15. We have $F \in \xi(m(p')) \Leftrightarrow m(p') \in F \Leftrightarrow p' \in m^{-1}(F) \Leftrightarrow m^{-1}(F) \in \xi'(p')$.

Theorems 14 and 15 show that there is a natural correspondence between state property systems and closure systems. Theorems 16 and 17 show that also the morphisms of both structures correspond. This indicates that the correspondence may be categorical. Indeed, we analyze the categorical aspect of this correspondence in detail in Aerts *et al.* (1999) and show that the category of state property systems and its morphisms and the category of closure spaces and continuous functions are equivalent categories.

The set of all testable properties is given by $\cup_{e \in \mathcal{E}} \mathcal{L}(e)$. Let us remark that *a priori* $\cup_{e \in \mathcal{E}} \mathcal{L}(e)$ is not a complete preorder set. This seems to contradict the results of earlier work. Indeed Piron (1976, 1989, 1990) and Aerts (1981, 1982, 1983) show that the set of all testable properties, hence $\cup_{e \in \mathcal{E}} \mathcal{L}(e)$, is a complete preorder set. We remark that in these earlier approaches equivalent properties are identified such that identified state property entities are considered: the complete preorder set is then a complete lattice, but this is not the origin of the problem that we want to point out here. We want to explain why in the earlier approaches completeness was derived for the set of all testable properties, while here we can only derive it for the set of testable properties connected to one definite experiment. First we remark that in the earlier approaches the complete preorder set was constructed by introducing explicitly all the mixed experiments. If we consider the mixed experiment $e(\mathscr{E})$ and the set of $e(\mathscr{E})$-testable properties $\mathscr{L}(e(\mathscr{E}))$, it can be shown that 'under a certain condition' the set of $e(\mathscr{E})$-testable properties contains all the other sets of $e(E)$-testable properties, where $E \subset \mathscr{E}$. This means that $\cup_{e\in\mathscr{E}} \mathscr{L}(e) = \mathscr{L}(e(\mathscr{E}))$. We have shown (Aerts, 1994) that the condition that implies this equality is a condition of 'distinguishable experiments.' This condition of 'distinguishable experiments' leading to the completeness of the set of all testable properties was unconsciously assumed in the already mentioned earlier approaches (Piron 1976, 1989, 1990; Aerts 1981, 1982, 1983). There it was taken for granted that an experiment, called test, question, or experimental project in Piron (1976, 1989, 1990) and Aerts (1981, 1982, 1983), that can be distinguished from all the others can be associated with each property (e.g., by labeling the test by means of the property). At first sight it seems indeed that it is always possible to do so. But in a formalism like the one we propose here, the properties as well as the experiments that can be used to test these properties are given from the start. It is against the 'rules of the game' to introduce new experiments for the properties just with the aim of being able to distinguish them from all the others. So we must conclude that the completeness can *a priori* only be shown for the set of testable properties connected to a definite experiment. Let us demonstrate the details of this situation in our formalism.

Proposition 15. Consider an entity $S(M(\mathscr{E}), M(\Sigma), M(X), \mathbb{O})$, and suppose that $e(E) \in M(\mathscr{E})$ is a mixed experiment and consider $A \subset O(e(E))$. We have

$$eig_{e(E)}(A) = \bigcap_{e\in E} eig_e(A \cap O(e)) \tag{84}$$

Proof. $p \in eig_{e(E)}(A) \Leftrightarrow O(e(E),p) \subset A \Leftrightarrow \cup_{e\in E} O(e, p) \subset A \Leftrightarrow O(e, p) \subset A\ \forall e \in E \Leftrightarrow O(e, p) \subset A \cap O(e)\ \forall e \in E \Leftrightarrow p \in eig_e(A \cap O(e))\ \forall e \in E \Leftrightarrow p \in \cap_{e\in E}\, eig_e(A \cap O(e))$.

Definition 28 (distinguishable experiment entity). Suppose that we have an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$. We say that two experiments $e, f \in \mathscr{E}$ are distinguishable iff $O(e) \cap O(f) = \emptyset$. We say that the entity S is a 'distinguishable experiment entity' iff $\forall e, f \in \mathscr{E}$ we have that e and f are distinguishable.

Two experiments f and g are distinguishable if they can be distinguished from each other by means of their outcomes. Let us explain intuitively in the spirit of Piron (1976, 1989, 1990) and Aerts (1981, 1982, 1983) why distinguishable experiments are necessary for the completeness of the set of testable properties. We will use the concept of test, question, or experimental project as introduced in Piron (1976, 1989, 1990) and Aerts (1981, 1982, 1983) without explicitly defining it again. The reader not acquainted with this concept can better skip this section and go on to just before the next proposition. There the intuitive reasoning that we will give now is repeated in the approach being developed in this paper.

Suppose that we consider a test $\alpha(f, A)$, consisting of performing the experiment f and giving the positive answer 'yes' if the outcome is in A, and a test $\alpha(g, B)$, consisting of performing the experiment g and giving a positive answer 'yes' if the outcome is in B. To 'prove' the completeness one introduces in Piron (1976, 1990) and Aerts (1981, 1982, 1983) the concept of 'product test,' and if $\alpha(f, A)$ tests whether the property $a(f, A)$ is actual and $\alpha(g, B)$ tests whether the property $a(g, B)$ is actual, then $\alpha(f, A) \cdot \alpha(g, B)$ tests whether an infimum of the properties $a(f, A)$ and $a(g, B)$ is actual. It is by requiring that the set of tests on the entity S contains all the product tests that the preorder set of testable properties becomes complete, because an infimum exists for each subset of properties. The product test is defined by means of the experiment $e(\{f, g\})$, and is given by $\alpha(e(\{f, g\}), A \cup B)$, consisting in performing the experiment $e(\{f, g\})$ and giving a positive answer 'yes' if the outcome is in $A \cup B$. We remark that, although the product test can always be defined, it only tests whether the two properties $a(f, A)$ and $a(g, B)$ are actual if f and g are distinguishable experiments. Indeed, suppose that f and g are not distinguishable; then $O(f) \cap O(g) \neq \emptyset$, which means that there is at least one outcome $x \in O(f) \cap O(g)$. Suppose that A does not contain this outcome, while B does; then it is possible that the entity S is in a state p such that e has as possible outcomes the set $A \cup \{x\}$, which is a state where $a(g, A)$ is not actual, and where g has as possible outcomes B. Then $e(\{f, g\})$ has as possible outcomes $A \cup B$, which means that in this state p the test $\alpha(e(\{f, g\}), A \cup B)$ gives with certainty a positive outcome. This shows that in this case of nondistinguishable experiments, $\alpha(e(\{f, g\}), A \cup B)$ does not test the actuality of the infimum of the properties $a(f, A)$ and $a(g, B)$.

Proposition 16. Suppose that we have an entity $S(M(\mathscr{E}), M(\Sigma), M(X), \mathbb{O})$. Suppose that we denote by $\mathscr{F}(e(\mathscr{E}))$ the collection of eigenstate sets of the experiment $e(\varepsilon)$. If all the experiments are distinguishable, then for $E \subset \mathscr{E}$ we have

$$\mathscr{F}(e(E)) \subset \mathscr{F}(e(\mathscr{E})) \tag{85}$$

Proof. Consider an arbitrary element $F \in \mathscr{F}(e(E))$. Then there exists $A \subset O(e(E))$ such that $F = eig_{e(E)}(A)$. Consider $A' = A \cup (\cup_{e \in \mathscr{E}, e \notin E} O(e))$; then we have $eig_{e(\mathscr{E})}(A') = eig_{e(E)}(A)$, which shows that $F \in \mathscr{F}(e(\mathscr{E}))$.

Theorem 18. Suppose that $S(M(\mathscr{E}), M(\Sigma), M(X), \mathbb{O})$ is a distinguishable experiment entity such that $e(\mathscr{E}) \in M(\mathscr{E})$. We then have

$$\bigcup_{e \in M(\mathscr{E})} \mathscr{F}(e) = \mathscr{F}(e(\mathscr{E})) \tag{86}$$

Proof. From Proposition 16 we have $\mathscr{F}(e) \subset \mathscr{F}(e(\mathscr{E}))$ for all $e \in M(\mathscr{E})$, which implies that $\cup_{e \in M(\mathscr{E})} \mathscr{F}(e) \subset \mathscr{F}(e(\mathscr{E}))$. Since $e(\mathscr{E}) \in M(\mathscr{E})$ we have $\mathscr{F}(e(\mathscr{E})) \subset \cup_{e \in M(\mathscr{E})} \mathscr{F}(e)$.

For such a distinguishable experiment entity we can also prove that the set of all testable properties is a complete preorder set.

Theorem 19. Suppose that $S(M(\mathscr{E}), M(\Sigma), M(X), \mathbb{O}, \mathscr{L}, \xi)$ is a distinguishable experiment entity such that $e(\mathscr{E}) \in M(\mathscr{E})$. We then have

$$\bigcup_{e \in M(\mathscr{E})} \mathscr{L}(e) = \mathscr{L}\ (e(\mathscr{E})) \tag{87}$$

and the set of testable properties $\cup_{e \in M(\mathscr{E})} \mathscr{L}(\mathrm{e})$ is a complete preorder set with a maximal element I such that $\kappa(\mathrm{I}) = M(\Sigma)$, and a minimal element 0 such that $\kappa(0) = \emptyset$.

Theorem 20. Suppose that $S(M(\mathscr{E}), M(\Sigma), M(X), \mathbb{O}, \mathscr{L}, \xi)$ is an identified distinguishable experiment entity such that $e(\mathscr{E}) \in M(\mathscr{E})$. Then the state property system $(\Sigma, \mathscr{L}(e(\mathscr{E})), \xi_{e(\mathscr{E})})$ describes the state property entity $S(\Sigma \mathscr{L}(e(\mathscr{E})), \xi_{e(\mathscr{E})})$, and $\mathscr{L}(e(\mathscr{E}))$ contains all testable properties of the entity.

We mention that all the calculations in the earlier approaches (Piron, 1976, 1989, 1990; Aerts, 1981, 1982, 1983) actually take place in the state property system $(\Sigma, \mathscr{L}(e(\mathscr{E})), \xi_{e(\mathscr{E})})$.

8. THE EIGENCLOSURE

We analyze now how for a state experiment outcome entity closure structures can be introduced in a natural way on the product set $\mathscr{E} \times \Sigma$ and the set of experiments $\mathscr{E}$.

Definition 29 (central eigenmap). Let us consider an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$. For $A \subset X$ we introduce

$$eig\colon \quad \mathscr{P}(X) \to \mathscr{P}(\mathscr{E} \times \Sigma), \qquad A \mapsto eig(A) \tag{88}$$

such that

$$(e, p) \in eig(A) \Leftrightarrow O(e, p) \subset A \tag{89}$$

Definition 30 (eigenmaps on the experiments). Let us consider an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$. For $p \in \Sigma$ we define a map eig_p that we call the eigenmap corresponding to the state p:

$$eig_p: \quad \mathscr{P}(O(p)) \to \mathscr{P}(\mathscr{E}), \qquad A \mapsto eig_p(A) \tag{90}$$

$$e \in eig_p(A) \Leftrightarrow O(e, p) \subset A \tag{91}$$

Proposition 17. Let us consider an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$ and the central eigenmap eig: $\mathscr{P}(X) \to \mathscr{P}(\mathscr{E} \times \Sigma)$, $A \mapsto eig(A)$; then for $A_i \subset X$ we have

$$eig(\cap_i A_i) = \cap_i eig(A_i) \tag{92}$$

Proof. $(e, p) \in eig(\cap_i A_i) \Leftrightarrow O(e, p) \subset \cap_i A_i \Leftrightarrow O(e, p) \subset A_i \; \forall_i \Leftrightarrow (e, p) \in eig(A_i) \; \forall_i \Leftrightarrow (e, p) \in \cap_i eig(A_i)$.

Proposition 18. The map eig_p introduced in Definition 30 satisfies the following properties:

$$eig_p(\emptyset) = \emptyset \tag{93}$$

$$eig_p(O(p)) = \mathscr{E} \tag{94}$$

$$eig_p(\cap_i A_i) = \cap_i eig_p(A_i) \tag{95}$$

Definition 31. Let us consider an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$ and the eigenmaps $eig_p, p \in \Sigma$. We denote the image of an eigenmap eig_p by $\mathscr{G}(p)$; in other words,

$$\mathscr{G}(p) = \{G \subset \mathscr{E} \mid \exists A \subset O(p), G = eig_p(A)\} \tag{96}$$

Definition 32. Let us consider an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$ and the central eigenmap eig. We denote the set of all images of eig by $\mathscr{Y}$, hence

$$\mathscr{Y} = \{Y \subset \mathscr{E} \times \Sigma \mid \exists A \subset X, Y = eig(A)\} \tag{97}$$

We have shown that $\mathscr{F}(e)$ is a closure system on Σ. In an analogous way we show that $\mathscr{Y}$ is a closure system on $\mathscr{E} \times \Sigma$ and $\mathscr{G}(p)$ is a closure system on $\mathscr{E}$.

Theorem 21. Let us consider an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$; then $\mathscr{Y}$ and $\mathscr{G}(p)$ are closure systems for every $e \in \mathscr{E}, p \in \Sigma$, respectively, on $\mathscr{E} \times \Sigma$, Σ, and $\mathscr{E}$.

Proof. We give the proof for $\mathscr{Y}$. Suppose that $Y_i \in \mathscr{Y}$. Then $\exists A_i \subset X$ such that $Y_i = eig(A_i)$. We have $(e, p) \in eig(\cap_i A_i) \Leftrightarrow (e, p) \in \cap_i eig(A_i) = \cap_i Y_i$.

Definition 33 (generating set). Suppose we have a set Z and $\mathscr{F}$ is the set of closed subsets corresponding to a closure operator cl on Z. The collection $\mathscr{B} \subset \mathscr{F}$ is a 'generating set' for $\mathscr{F}$ iff for each subset $F \in \mathscr{F}$ we have a family $B_i \in \mathscr{B}$ such that $F = \cap_i B_i$.

Proposition 19. Suppose we have a set Z equipped with a closure cl and $\mathcal{B}$ is a generating set for the set of closed subsets $\mathcal{F}$. Then for an arbitrary subset $K \subset Z$ we have

$$cl(K) = \bigcap_{K \subset B, B \in \mathcal{B}} B \tag{98}$$

Proof. We know that $cl(K) = \cap_{K \subset F, F \in \mathcal{F}} F$. Because $\mathcal{B}$ is a generating set for $\mathcal{F}$ we have $F = \cap_{F \subset B, B \in \mathcal{B}} B$. Hence $cl(K) = \cap_{k \subset F}(\cap_{F \subset B} B) = \cap_{K \subset B, B \in \mathcal{B}} B$.

On the set of states Σ we have a collection of closure systems $\mathcal{F}(e)$, $e \in \mathcal{E}$. It is easy to show that they generate a global closure system on Σ.

Theorem 22. Let us consider an entity $S(\mathcal{E}, \Sigma, X, \mathcal{O})$ and the set of eigenmaps $\{eig_e | e \in \mathcal{E}\}$ and corresponding closure systems $\mathcal{F}(e)$. Put $\mathcal{A} = \cup_{e \in \mathcal{E}} \mathcal{F}(e)$, and consider

$$\mathcal{F} = \{\cap_i A_i \mid A_i \in \mathcal{A}\} \tag{99}$$

Then $\mathcal{F}$ is a closure system on Σ generated by $\mathcal{A}$.

Proof. Consider $F_i \in \mathcal{F}$. Then there exist $A_{ij} \in \mathcal{A}$ such that $F_i = \cap_j A_{ij}$. We now have $\cap_i F_i = \cap_i \cap_j A_{ij}$ which shows that $\cap_i F_i \in \mathcal{F}$.

In an analogous way the set of closure systems $\mathcal{G}(p)$ on $\mathcal{E}$ generates a global closure system on $\mathcal{E}$.

Theorem 23. Let us consider an entity $S(\mathcal{E}, \Sigma, X, \mathcal{O})$ and the set of eigenmaps $\{eig_p | p \in \Sigma\}$ and corresponding closure systems $\mathcal{G}(p)$. Put $\mathcal{C} = \cup_{p \in \Sigma} \mathcal{G}(p)$, and consider.

$$\mathcal{G} = \{\cap_i C_i | C_i \in \mathcal{C}\} \tag{100}$$

Then $\mathcal{G}$ is a closure system on $\mathcal{E}$ generated by $\mathcal{C}$.

Definition 34. Let us consider an entity $S(\mathcal{E}, \Sigma, X, \mathcal{O})$; we shall call $\mathcal{Y}$, $\mathcal{F}$, and $\mathcal{G}$, respectively, the central eigen-, the state eigen-, and the experiment eigenclosure system and denote them from now on $\mathcal{F}_{eig}$, $\mathcal{G}_{eig}$, and $\mathcal{Y}_{eig}$. To make notations not to heavy, we will denote the closure operator for each of the closure system by cl_{eig}. Hence

$$\begin{aligned} cl_{eig}&: \mathcal{P}(\mathcal{E} \times \Sigma) \to \mathcal{Y}_{eig} \subset \mathcal{P}(\mathcal{E} \times \Sigma), & K &\mapsto cl_{eig}(K) = \bigcap_{K \subset Y, Y \in \mathcal{Y}} Y \\ cl_{eig}&: \mathcal{P}(\Sigma) \to \mathcal{F}_{eig} \subset \mathcal{P}(\Sigma), & K &\mapsto cl_{eig}(K) = \bigcap_{K \subset F, F \in \mathcal{F}} F \\ cl_{eig}&: \mathcal{P}(\mathcal{E}) \to \mathcal{G}_{eig} \subset \mathcal{P}(\mathcal{E}), & K &\mapsto cl_{eig}(K) = \bigcap_{K \subset G, G \in \mathcal{G}} G \end{aligned} \tag{101}$$

We could ask ourselves now what the relation is between the closures $\mathcal{Y}_{eig}$,

$\mathcal{F}_{eig}$, and $\mathcal{G}_{eig}$. Could it be that the state eigenclosure and the experiment eigenclosure are in some way 'traces' of the central eigenclosure? To see whether this is the case, let us introduce the following.

Definition 35. Let us consider an entity $S(\mathcal{E}, \Sigma, X, \mathcal{O})$ and the central eigenclosure $\mathcal{Y}_{eig}$. For $Y \in \mathcal{Y}_{eig}$ we introduce

$$Y_{\text{state}} = \{p \in \Sigma \mid \forall\ e \in \mathcal{E}, (e, p) \in Y\} \tag{102}$$

We then define

$$\mathcal{Y}_{eig}(\text{state}) = \{Y_{\text{state}} \mid Y \in \mathcal{Y}_{eig}\} \tag{103}$$

Proposition 20. $\mathcal{Y}_{eig}$(state) is a closure system on the set of states Σ.

Proof. Consider $\mathcal{E} \times \Sigma \in \mathcal{Y}_{eig}$; then $(\mathcal{E} \times \Sigma)_{\text{state}} = \Sigma$, which shows that $\Sigma \in \mathcal{Y}_{eig}(\text{state})$. Obviously $\emptyset \in \mathcal{Y}_{eig}(\text{state})$. Consider now $Z_i \in \mathcal{Y}_{eig}(\text{state})$, which means that there exists $Y_i \in \mathcal{Y}_{eig}$ such that $Z_i = (Y_i)_{\text{state}}$. Consider $(\cap_i Y_i)_{\text{state}}$. We have $p \in (\cap_i Y_i)_{\text{state}} \Leftrightarrow \forall e \in \mathcal{E}, (e, p) \in \cap_i Y_i \Leftrightarrow \forall e \in \mathcal{E}, \forall i, (e, p) \in Y_i, \Leftrightarrow i, p \in (Y_i)_{\text{state}} \Leftrightarrow p \in \cap_i (Y_i)_{\text{state}}$.

In the example of Section 14 we show that in general $\mathcal{F}_{eig}$ is not equal to $\mathcal{Y}_{eig}$(state), but we can prove the equality for distinguishable experiment entities.

Theorem 24. Let us consider a distinguishable experiment entity $S(\mathcal{E}, \Sigma, X, \mathcal{O})$ with central eigenclosure system $\mathcal{Y}_{eig}$ and state eigenclosure system $\mathcal{F}_{eig}$. Then we have

$$\mathcal{F}_{eig} = \mathcal{Y}_{eig}\ (\text{state}) \tag{104}$$

Proof. It is enough to show that each element of the generating set $\mathcal{A}$ of the state eigenclosure system $\mathcal{F}_{eig}$ also belongs to $\mathcal{Y}_{eig}$(state). Suppose that $A \subset O(e)$ and hence $eig_e(A) \in \mathcal{A}$. Consider now the set $B = \cup_{f \in \varepsilon, f \neq e} O(f) \cup A$. Remark first that $O(e, p) \subset A \Rightarrow O(f, p) \subset B, \forall f \in \mathcal{E}$. Let us show that because S is a distinguishable experiment entity, we also have the inverse implication. Let us remark that $B \cap O(g) = \cup_{f \in \varepsilon, f \neq e} [(O(f) \cap O(g)) \cup (A \cap O(g))]$, which shows that $B \cap O(e) = A$. Let us now consider $p \in eig(B)_{\text{state}} \Rightarrow \forall f \in \mathcal{E}, O(f, p) \subset B \Rightarrow O(e, p) \subset B \cap O(e) = A \Rightarrow p \in eig_e(A)$. So $eig_e(A) = (eig(B))_{\text{state}} \in \mathcal{Y}(\text{state})$. For the converse suppose $eig(A)_{\text{state}} \in \mathcal{Y}(\text{state})$, where $A \subset X$; then $eig(A)_{\text{state}} = \cap_{e \in \mathcal{E}} eig_e(A \cap O(e))$.

To finish this section on the eigenclosures, we show that there is also a very natural closure structure on the set of outcomes on an entity.

Definition 36. Let us consider on entity $S(\mathcal{E}, \Sigma, X, \mathcal{O})$. For $A \subset X$ we define

$$cl(A) = \bigcap_{O(e,p) \subset A^c} O(e, p)^C \tag{105}$$

Theorem 25. Let us consider an entity $S(\mathcal{E}, \Sigma, X, \mathcal{O})$ and the map cl

introduced in Definition 36. Then cl is a closure on X. All the $O(e, p)$ are open sets for this closure and

$$O(e, p) \subset cl(A)^C \Leftrightarrow O(e, p) \subset A^C \tag{106}$$

$$eig(A) = eig(int(A)) \tag{107}$$

Proof. Clearly $A \subset cl(A)$. If $A \subset B$, then $cl(A) \subset cl(B)$. We have that $cl\ (\emptyset) = \emptyset$. We have to show now that $cl(cl(A)) \subset cl(A)$. Let us first prove that $O(e, p) \subset cl(A)^C \Leftrightarrow O(e, p) \subset A^C$ because from this there follows immediately the closure property that we are left to prove. Suppose that $O(e, p) \subset cl(A)^C$; then $O(e, p) \subset \cup_{O(f, q) \subset A^C} O(f, q) \subset A^C$. On the other hand, suppose that $O(e, p) \subset A^C$; then $cl(A) \subset \cup_{O(f, q) \subset A^C} O(f, q) \subset O(e, p)^C$, which implies that $O(e, p) \subset cl(A)^C$. Now we have $cl(cl(A)) = \cap_{O(e,p) \subset cl(A)^C} O(e, p)^C = \cap_{O(e,p) \subset A^C} O(e, p)^C = cl(A)$.

9. ORTHOGONALITY AND ORTHOCLOSURE

The orthogonality relations give rise to a closure in a natural way.

Proposition 21. Consider a set Z equipped with an orthogonality relation $\perp$, and define for $K \subset Z$ the set $K^\perp = \{p \mid p \perp q, q \in K\}$, and

$$cl(K) = (K^\perp)^\perp \tag{108}$$

Then cl is a closure operator that we shall call the orthoclosure operator connected to $\perp$.

Proof. See Birkhoff (1978).

Proposition 22. Let us denote the collection of orthoclosed subsets by $\mathcal{Y}_{orth}$; then it can easily be shown that this closure system is orthocomplemented, which means that the map $^\perp$: $\mathcal{Y}_{orth} \rightarrow \mathcal{Y}_{orth}$ satisfies

$$K \subset L \Rightarrow L^\perp \subset K^\perp, \qquad K^{\perp\perp} = K, \qquad K \cap K^\perp = \emptyset \tag{109}$$

Proposition 23. The following formulas are satisfied in $\mathcal{Y}_{orth}$ for $Y_i \in \mathcal{Y}_{orth}$:

$$(\cap_i Y_i)^\perp = cl(\cup_i Y_i^\perp)$$

$$(\cup_i Y_i)^\perp = \bigcap_i Y_i^\perp \tag{110}$$

$$cl(Y \cup Y^\perp) = Z$$

Proof. Let $Y \in \mathcal{Y}_{orth}$: (1) $cl_{orth}\ (\cup_i Y_i^\perp) \subset Y \Leftrightarrow \cup_i Y_i^\perp \subset Y \Leftrightarrow Y_i^\perp \subset Y\ \forall i \Leftrightarrow Y^\perp \subset Y_i\ \forall i \Leftrightarrow Y^\perp \subset \cap_i Y_i \Leftrightarrow (\cap_i Y_i)^\perp \subset Y$. From this it follows that $cl_{orth}\ (\cup_i Y_i^\perp) = (\cap_i Y_i)^\perp$. (2) $Y \subset \cap_i Y_i^\perp \Leftrightarrow Y_i \subset Y^\perp\ \forall i \Leftrightarrow \cup_i Y_i \subset$

$Y^{\perp} \Leftrightarrow cl_{orth}(\cup_i Y_i) \subset Y^{\perp} \Leftrightarrow Y \subset (\cup_i Y_i)^{\perp}$. From this it follows that $(\cup_i Y_i)^{\perp} = \cap_i Y_i^{\perp}$. (3) $cl_{orth}(Y \cup Y^{\perp}) = (Y^{\perp} \cap Y)^{\perp} = \emptyset^{\perp} = Z$.

An orthoclosure system has a simple generating set of elements.

Theorem 26. The set $\mathcal{B} = \{\{p\}^{\perp} | \, p \in Z\}$ is a generating set for the set of orthoclosure system $\mathcal{Y}_{orth}$.

Proof. Consider any element $Y \in \mathcal{Y}$. We have $Y^{\perp} = \cup_{p \in Y^{\perp}} \{p\}$, and hence $Y = Y^{\perp\perp} = \cap_{p \in Y^{\perp}} \{p\}^{\perp}$.

Definition 37. Let us consider an entity $S(\mathcal{E}, \Sigma, X, \mathcal{O})$. We have defined orthogonality relations on $\mathcal{E} \times \Sigma$, on Σ, and on $\mathcal{E}$. We will call the orthoclosure systems related to these orthogonality relations, the central ortho-, the state ortho-, and the experiment orthoclosure systems and denote them respectively by $\mathcal{Y}_{orth}$, $\mathcal{F}_{orth}$, and $\mathcal{G}_{orth}$.

We can prove the following surprising result:

Theorem 27. Let us consider an entity $S(\mathcal{E}, \Sigma, X, \mathcal{O})$ and the eigenclosure systems $\mathcal{Y}_{eig}$, $\mathcal{F}_{eig}(e)$, and $\mathcal{G}_{eig}(p)$ and the orthoclosure systems $\mathcal{Y}_{orth}$, $\mathcal{F}_{orth}(e)$, and $\mathcal{G}_{orth}(p)$. We have

$$\mathcal{Y}_{orth} \subset \mathcal{Y}_{eig}, \qquad \mathcal{F}_{orth}(e) \subset \mathcal{F}_{eig}(e), \qquad \mathcal{G}_{orth}(p) \subset \mathcal{G}_{eig}(p) \tag{111}$$

Proof. Consider $Y \in \mathcal{Y}_{orth}$ and consider $A = \cup_{(e,p) \in Y^{\perp}} O(e, p)$. Since $Y = (Y^{\perp})^{\perp}$, we have $(f, q) \in Y \Leftrightarrow O(f, q) \cap O(e, p) = \emptyset \ \forall (e, p) \in Y^{\perp} \Leftrightarrow O(f, q) \cap A = \emptyset \Leftrightarrow O(f, q) \subset A^C \Leftrightarrow (f, q) \in eig(A^C)$. This shows that $Y \in \mathcal{Y}_{eig}$. Consider now $F \in \mathcal{F}_{orth}(e)$ and $B = \cup_{p \in F^{\perp_e}} O(e, p)$. Since $F = (F^{\perp_e})^{\perp_e}$ we have $q \in F \Leftrightarrow O(e, q) \cap O(e, p) = \emptyset, \forall p \in F^{\perp_e} \Leftrightarrow O(e, q) \cap B = \emptyset \Leftrightarrow O(e, q) \subset B^C \Leftrightarrow q \in eig_e(B^C)$. This shows that $F \in \mathcal{F}_{eig}(e)$.

10. OUTCOME, EXPERIMENT, AND STATE DETERMINATION AND THE FIRST SEPARATION AXIOM

In this section we will show that the traditional topological separation axioms are connected to physically well interpretable properties of the considered entities. Instead of introducing these properties as axioms, we choose to use them as characterizations of types of entities.

Definition 38. Let us consider an entity $S(\mathcal{E}, \Sigma, X, \mathcal{O})$. We say that the entity is 'outcome determined' iff $O(e, p) = O(f, q) \Rightarrow (e, p) = (f, q)$.

Definition 39. Consider a set W with a closure operator cl. We say that cl satisfies the T_0 separation axiom iff for $w, v \in W$ we have $cl(w) = cl(v) \Rightarrow w = v$.

Proposition 24. Let us consider an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$. Suppose that cl_{eig} is the eigenclosure operator on $\mathscr{E} \times \Sigma$. We have

$$cl_{eig}(\{(e, p)\}) = eig(O(e, p)) \tag{112}$$

Proof. Since $\{(e, p)\} \subset eig(O(e, p))$ we have $cl_{eig}(\{(e, p)\}) \subset eig(O(e, p))$ because $cl_{eig}(\{(e, p)\})$ is the smallest element of $\mathscr{Y}_{eig}$ that contains $\{(e, p)\}$. Let us prove now that $eig(O(e, p)) \subset cl_{eig}(\{(e, p)\})$. Since $cl_{eig}(\{(e, p)\}) \in \mathscr{Y}_{eig}$ there exists a set $A \subset X$ such that $cl_{eig}(\{(e, p)\}) = eig(A)$. From this it follows that $(e, p) \in eig(A)$, or $O(e, p) \subset A$. This implies that $eig(O(e, p)) \subset eig(A)$, and hence we have shown that $eig(O(e, p)) \subset cl_{eig}(\{(e, p)\})$. As a consequence $cl_{eig}(\{(e, p)\}) = eig(O(e, p))$.

Theorem 28. Let us consider an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$. The entity S is 'outcome determined' iff the central eigenclosure operator satisfies the T_0 separation axiom.

Proof. Suppose that the entity is 'outcome determined' and consider $(e, p), (f, q) \in \mathscr{E} \times \Sigma$ such that $cl_{eig}(\{(e, p)\}) = cl_{eig}(\{(f, q)\})$. From the foregoing theorem then it follows that $eig(O(e, p)) = eig((O(f, q))$. This means that $(e, p) \in eig(O(f, q))$, or $O(e, p) \subset O(f, q)$ and also $(f, q) \in eig(O(e, p))$ and hence $O(f, q) \subset O(e, p)$. From this it follows that $O(e, p) = O(f, q)$ and hence, since the entity is 'outcome determined,' we have $(e, p) = (f, q)$. This shows that cl_{eig} is T_0. Suppose now that the central eigenclosure operator is T_0, and consider (e, p) and (f, q) such that $O(e, p) = O(f, q)$. Then $eig(O(e, p)) = eig(O(f, q))$ and hence $cl_{eig}(\{(e, p)\}) = cl_{eig}(\{(f, q)\})$. From this follows that $(e, p) = (f, q)$, and hence we have shown that the entity is 'outcome determined.'

Let us investigate now the eigenclosure on the set of states. We can characterize the closure of singletons in the following way:

Proposition 25. Let us consider an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$. Suppose that cl_{eig} is the state eigenclosure operator on Σ. We have

$$cl_{eig}(\{p\}) = \bigcap_{e \in \mathscr{E}} eig_e(O(e, p)) \tag{113}$$

Proof. Since $\{p\} \subset eig_e(O(e, p))\ \forall e \in \mathscr{E}$ we have $\{p\} \subset \cap_{e \in \mathscr{E}} eig_e(O(e, p))$. This shows that $cl_{\ eig}(\{p\}) \subset \cap_{e \in \mathscr{E}} eig_e(O(e, p))$. Let us now prove the inverse inclusion. Since $cl_{eig}(\{p\}) \in \mathscr{F}_{eig}$ there exists for $e \in \mathscr{E}$, $A(e) \subset O(e)$ such that $cl_{eig}(\{p\}) = \cap_{e \in \mathscr{E}} eig_e(A(e))$. We have $\{p\} \subset \cap_{e \in \mathscr{E}} eig_e(A(e))$ and hence $p \in eig_e(A(e))\ \forall e \in \mathscr{E}$. But this implies that $O(e, p) \subset A(e)\ \forall e \in \mathscr{E}$, which in turn implies that $eig_e(O(e, p)) \subset eig_e(A(e))\ \forall e \in \mathscr{E}$. As a consequence $\cap_{e \in \mathscr{E}} eig_e\ (O(e, p)) \subset \cap_{c \leftarrow \mathscr{E}} eig_e\ A(e))$ which shows that $\cap_{e \in \mathscr{E}} eig_e(O(e, p)) \subset cl_{eig}(\{p\})$.

Definition 40. Let us consider an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$. We say that the entity is 'state determined' iff $O(e, p) = O(e, q)\ \forall e \in \mathscr{E} \Rightarrow p = q$.

Theorem 29. Let us consider an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$. The entity S is 'state determined' iff the state eigenclosure operator satisfies the T_0 separation axiom.

Proof. Suppose that the entity is 'state determined' and consider $p, q \in \Sigma$ such that $cl_{eig}(\{p\}) = cl_{eig}(\{q\})$. This means that $\cap_{e \in \mathscr{E}}\ eig_e(O(e, p)) = \cap_{e \in \mathscr{E}}\ eig_e(O(e, q))$. From this it follows that $p \in eig_e(O(e, q))\ \forall e \in \mathscr{E}$. Hence $O(e, p) \subset O(e, q)\ \forall e \in \mathscr{E}$. In an analogous way we show that $O(e, q) \subset O(e, p)\ \forall e \in \mathscr{E}$, which proves that $O(e, p) = O(e, q)\ \forall e \in \mathscr{E}$. Since the entity is state determined we have as a consequence that $p = q$. This proves that the state eigenclosure operator satisfies the T_0 separation axiom. Suppose now that the state closure operator is T_0 and consider $p, q \in \Sigma$ such that $O(e, p) = O(e, q)\ \forall e \in \mathscr{E}$. Then $\cap_{e \in \mathscr{E}}\ O(e, p) = \cap_{e \in \mathscr{E}}\ O(e, q)$, and hence $cl_{eig}(\{p\}) = cl_{eig}(\{q\})$. From this it follows that $p = q$ and hence we have shown that the entity is 'experiment determined.'

By symmetry we can formulate analogous properties for 'experiment-determined' entities.

Definition 41. Let us consider an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$. We say that the entity is 'experiment determined' iff $O(e, p) = O(f, p)\ \forall p \in \Sigma \Rightarrow e = f$.

Theorem 30. Let us consider an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$. The entity S is 'experiment determined' iff the experiment eigenclosure operator satisfies the T_0 separation axiom.

11. ATOMIC ENTITIES AND THE SECOND SEPARATION AXIOM

The second topological separation axiom is also connected to a property that we can easily interpret physically.

Definition 42. Let us consider an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$. We say that the entity is 'central atomic' iff $(e, p) < (f, q) \Rightarrow (e, p) = (f, q)$.

Definition 43. Consider a set W with a closure operator cl. We say that cl satisfies the T_1 separation axiom iff for $w \in W$ we have $cl(\{w\}) = \{w\}$.

Proposition 26. Let us consider an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$. The entity S is 'central atomic' iff the central eigenclosure operator satisfies the T_1 separation axiom.

Proof. Suppose that the entity is 'central atomic' and consider $(e, p) \in \mathscr{E} \times \Sigma$. Suppose that $(f, q) \in cl_{eig}(\{e, p\})$; then $(f, q) \in eig(O(e, p))$. This means that $O(f, q) \subset O(e, p)$ and hence $(f, q) < (e, p)$. But from this it follows that $(f, q) = (e, p)$. So we have shown that $cl_{eig}(\{(e, p)\})$ contains no other elements than (e, p) and hence $cl_{eig}(\{(e, p)\}) = \{(e, p)\}$. On the other hand, suppose now that the central eigenclosure operator is T_1, and consider $(e, p) < (f, q)$. We then have $O(e, p) \subset O(f, q)$ and hence $eig(O(e, p)) \subset eig(O(f, q))$, which implies that $\{(e, p)\} = cl_{eig}(\{(e, p)\}) \subset cl_{eig}(\{f, q)\}) = \{(f, q)\}$. This proves that $(e, p) = (f, q)$.

Definition 44. Let us consider an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$. We say that the entity is 'state atomic' iff $p < q \Rightarrow p = q$.

Theorem 31. Let us consider an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$. The entity S is 'state atomic' iff the state eigenclosure operator satisfies the T_1 separation axiom.

Proof. Suppose that the entity is 'state atomic' and consider $p \in \Sigma$. Suppose that $q \in cl_{eig}(\{p\})$; then $q \in \cap_{e \in \mathscr{E}} eig(O(e, p))$, and hence $q \in eig(O(e, p))\ \forall e \in \mathscr{E}$. This means that $O(e, q) \subset O(e, p)\ \forall e \in \mathscr{E}$ and hence $q < p$. But from this it follows that $q = p$. So we have shown that $cl_{eig}(\{(p)\})$ contains no other elements than p and hence $cl_{eig}(\{p)\}) = \{p\}$. On the other hand, suppose now that the state closure operator is T_1, and consider $p < q$. We then have $O(e, p) \subset O(e, q)\ \forall e \in \mathscr{E}$ and hence $\cap_{e \in \mathscr{E}} eig(O(e, p)) \subset \cap_{e \in \mathscr{E}} eig(O(e, q))$, which implies that $\{p\} = cl_{eig}(\{p\}) \subset cl_{eig}(\{q\}) = \{q\}$. This proves that $p = q$.

Again for reasons of symmetry we have the corresponding theorem for 'experiment atomic' entities.

Definition 45. Let us consider an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$. We say that the entity is 'experiment atomic' iff $e < f \Rightarrow e = f$.

Theorem 32. Let us consider an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$. The entity S is 'experiment atomic' iff the experiment eigenclosure operator satisfies the T_1 separation axiom.

The following is now merely a reformulation of $T_1 \Rightarrow T_0$:

Theorem 33. Let us consider an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$. If the entity is 'central atomic,' then it is 'outcome determined.' If it is 'state atomic,' then it is 'state determined,' and if it is 'experiment atomic,' it is 'experiment determined.'

12. D-CLASSICAL ENTITIES

We want to study entities with special properties that make them 'more classical.' Since the word 'classical' is used in so many different meanings in different approaches, we will choose to introduce new names for these special properties.

Definition 46 (d-classical entity). Let us consider an entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$. We say that S is a 'd-classical' entity ('d' for deterministic) iff $\forall e \in \mathscr{E}$, $p \in \Sigma$ we have that $O(e, p)$ is a singleton, which we denote $O(e, p) = \{x(e, p)\}$.

Theorem 34. Let us consider a d-classical entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$.

1. For $p \in \Sigma$ and $e \in \mathscr{E}$, we have that p is always an eigenstate for the experiment e with eigenoutcome $x(e, p)$.

2. If $p, q \in \Sigma$ and $e, f \in \mathscr{E}$ such that $p < q$ and $e < f$, then $q < p$ and $f < e$, and hence $p \approx q$ and $e \approx f$.

Proof. Follows immediately from the definitions.

Theorem 35. Let us consider a d-classical entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$. Suppose that $p, q \in \Sigma$ and $e, f \in \mathscr{E}$; then we have that $p \approx q$ or $p \perp q$, and $e \approx f$ or $e \perp f$.

Proof. Consider $p, q \in \Sigma$ and suppose that $p \not\approx q$. This means that $p \not< q$ and $q \not< p$. Then there exists at least one experiment $e \in \mathscr{E}$ such that $O(e, p) \not\subset O(e, q)$. We have $O(e, p) = \{x(e, p)\}$ and $O(e, q) = \{x(e, q)\}$. Hence $O(e, p) \cap O(e, q) = \emptyset$, which shows that $p \perp q$. This shows that nonequivalent states are orthogonal. In an analogous way we show that experiments are equivalent or orthogonal for a d-classical entity.

Theorem 36. Let us consider a d-classical entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$. If the entity is 'outcome determined,' then it is 'central atomic.' If the entity is 'state determined,' then it is 'state atomic,' and if the entity is 'experiment determined,' then it is 'experiment atomic.'

Proof. Suppose that the entity is outcome determined. Consider $(e, p) < (f, g)$; then we have $O(e, p) \subset O(f, q)$ and hence $\{x(e, p)\} \subset \{x(f, q)\}$. This implies that $\{x(e, p)\} = \{x(f, q)\}$ and hence, since the entity is 'outcome determined,' we have $(e, p) = (f, q)$. So we have proved that the entity is central atomic. In an analogous way one proves the two other implications.

Let us now study the closures for d-classical entities.

Theorem 37. Let us consider a d-classical entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$. We have for $A \subset X$

$$eig(A) = \{(e, p) \mid x(e, p) \in A\} = x^{-1}(A) \tag{114}$$

$$eig(A^C) = eig(A)^C = eig(A)^\perp \tag{115}$$

$$\mathscr{Y}_{eig} = \mathscr{Y}_{orth} \tag{116}$$

Proof. We have $(e, p) \in eig(A) \Leftrightarrow O(e, p) \subset A$. Since $O(e, p) = \{x(e, p)\}$ we have $(e, p) \in eig(A) \Leftrightarrow x(e, p) \in A$. This shows that $eig(A) = \{(e, p) \mid x(e, p) \in A\}$. Consider now $(e, p) \in eig(A^C)$; then $x(e, p) \in$

A^C, and hence $x(e, p) \notin A$, which implies that $(e, p) \notin eig(A)$ or $(e, p) \in eig(A)^C$. This shows that $eig(A^C) \subset eig(A)^C$. Consider now $(e, p) \in eig(A)^C$, which means that $(e, p) \notin eig(A)$ and hence $x(e, p) \notin A$. Consider now an arbitrary $(f, q) \in eig(A)$, i.e., $x(f, q) \in A$. This means that $O(e, p) \cap O(f, q) = \{x(e, p)\} \cap \{x(f, q)\} = \emptyset$. As a consequence $(e, p) \in eig(A)^{\perp}$. This shows that $eig(A)^C \subset eig(A)^{\perp}$. Consider now $(e, p) \in eig(A)^{\perp}$. This means that $(e, p) \perp (f, q)$ for all $(f, q) \in eig(A)$. Hence $x(e, p) \in A^C$, which shows that $(e, p) \in eig(A^C)$. Hence we have shown that $eig(A)^{\perp} \subset eig(A^C)$. Let us prove now that $\mathcal{Y}_{eig} = \mathcal{Y}_{orth}$. We already have $\mathcal{Y}_{orth} \subset \mathcal{Y}_{eig}$ such that we only have to prove the inverse inclusion. If we remark that $eig(A) = eig(A^C)^{\perp}$, it follows that $eig(A) \in \mathcal{Y}_{orth}$.

Theorem 38. Let us consider a d-classical entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$. We have for $A \subset O(e)$

$$eig_e(A) = \{p \mid x(e, p) \in A\} \tag{117}$$

$$eig_e(A^C) = eig_e(A)^C = eig_e(A)^{\perp_e} \tag{118}$$

$$\mathcal{F}_{eig}(e) = \mathcal{F}_{orth}(e) \tag{119}$$

Theorem 39. Let us consider a d-classical entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$. We have for $A \subset O(p)$

$$eig_p(A) = \{e \mid x(e, p) \in A\} \tag{120}$$

$$eig_p(A^C) = eig_p(A)^C = eig_p(A)^{\perp_p} \tag{121}$$

$$\mathcal{G}_{eig}(p) = \mathcal{G}_{orth}(p) \tag{122}$$

A d-classical entity is a trivial type of probabilistic entity.

Theorem 40. Let us consider a *d*-classical entity $S(\mathscr{E}, \Sigma, X, \mathbb{O})$. It is a probabilistic entity where the probabilities are defined as follows:

$$\mu: \quad \mathscr{E} \times \Sigma \times X \to [0, 1], \qquad (e, p, y) \mapsto \mu(e, p, y) \tag{123}$$

where $\mu(e, p, y) = 0$ if $y \neq x(e, p)$ and $\mu(e, p, x(e, p)) = 1$.

13. SUBENTITIES AND MORPHISMS

The concept of subentity should be clearly defined. When will we decide that a certain ‘piece’ of an entity is a subentity? Let us consider two entities S and S' with sets of states Σ and Σ', sets of experiments $\mathscr{E}$ and $\mathscr{E}'$, and sets of outcomes X and X'. If S is to be a part of S', it is plausible to demand that if the entity S' is in a certain state p', then the entity S, as part of S', is in a well defined state $m(p')$. This defines a function:

$$m: \quad \Sigma' \to \Sigma \quad p' \mapsto m(p') \tag{124}$$

which is surjective—each state of the subentity S corresponds to at least one state of the entity S'—but not necessarily injective—different states of the entity S' can give rise to the same state of the subentity S. This function formalizes: "If S is a subentity of S', then the mode of being of S' determines that of S."

Second, if S is to be a part of S', this should imply that to each experiment e that can be performed on S there corresponds an experiment $n(e)$ that can be performed on S'. This again defines a function

$$n: \quad \mathscr{E} \to \mathscr{E}', \quad e \mapsto n(e) \tag{125}$$

which is injective—if experiments are different when they are performed on the subentity S, they are also different when they are performed on the entity S'—but not necessarily surjective—there can be experiments that can be performed on the entity S' that have no counterpart on the subentity S.

We have to express now that S is really a subentity of S by means of a requirement on the way experiments act on states and outcomes occur. This is again a requirement of 'covariance': reality does not depend on whether we represent a big piece of it by means of the entity S' or a smaller subpiece of it by means of the subentity S. More concretely, we express this requirement of covariance in the following way: if we perform an experiment e on the entity S in a state $m(p')$, where p' is a state of S', then the outcome $x(e, m(p'))$ occurs iff one specific outcome $x'(n(e), p')$ occurs after the performance of the experiment $n(e)$ on the entity S' in state p'. This requirement again defines a function

$$l: \quad X \to X', \quad x \mapsto l(x) \tag{126}$$

that is such that considering a state $p' \in \Sigma'$ and an experiment $e \in \mathscr{E}$, each outcome $x' \in O(n(e), p')$ corresponds to an outcome $x \in O(e, m(p'))$, such that $l(x) = x'$. The interpretation is that x occurs for e, S being in state $m(p')$ iff $l(x)$ occurs for $n(e)$, S' being in state p'. Since only one outcome occurs at once, this implies that the function l is injective, and $O(e, m(p'))$ is surjectively mapped onto $O(n(e), p')$ by l.

We have now introduced all elements to present a definition:

Definition 47 (subentities). Suppose that $S(\mathscr{E}, \Sigma, X, \mathbb{O})$ and $S'(\mathscr{E}', \Sigma', X', \mathbb{O}')$ are two entities. We say that S is a subentity of S' iff there exist a surjective function

$$m: \quad \Sigma' \to \Sigma, \quad p' \mapsto m(p') \tag{127}$$

an injective function

$$n: \quad \mathscr{E} \to \mathscr{E}', \quad e \mapsto n(e) \tag{128}$$

and an injective function

$$l: \quad X \to X', \quad x \mapsto l(x) \tag{129}$$

that for each $p' \in \Sigma'$ and $e \in \mathscr{E}$ maps $O(e, m(p'))$ surjectively on $O(n(e), p')$ such that:

(i) If the entity S' is in state p', then the entity S is in state $m(p')$.

(ii) If the experiment e is performed on the entity S, then the experiment $n(e)$ is performed on the entity S'.

(iii) Considering a state p' of S' and an experiment $e \in \mathscr{E}$, then the outcome $x \in O(e, m(p'))$ occurs for e being performed on S in state $m(p')$ iff the outcome $l(x)$ occurs for $n(e)$ being performed on S' in state p'.

Proposition 27. Suppose that $S(\mathscr{E}, \Sigma, X, \mathbb{O})$ and $S'(\mathscr{E}', \Sigma', X', \mathbb{O}')$ are two entities such that S is a subentity of S', and m, n, and l are the connecting functions related to S and S'. If $x, y \in X$, $e, f \in \mathscr{E}$, and $p', q' \in \Sigma'$ we have

$$x \perp y \Rightarrow l(x) \perp l(y) \tag{130}$$

$$p' < q' \Rightarrow m(p') < m(q') \tag{131}$$

$$e \perp f \Rightarrow n(e) \perp n(f) \tag{132}$$

$$(e, m(p')) < (f, m(q')) \Leftrightarrow (n(e), p') < (n(f), q') \tag{133}$$

$$(e, m(p')) \perp (f, m(q')) \Leftrightarrow (n(e), p') \perp (n(f), q') \tag{134}$$

Proof. Suppose that $x \perp y$; then there exists $e \in \mathscr{E}$ and $p \in \Sigma$ such that $x, y \in O(e, p)$ and $x \neq y$. Since m is surjective we have a $p' \in \Sigma'$ such that $p = m(p')$. This means that $x, y \in O(e, m(p'))$ and hence $l(x), l(y) \in O(n(e), p')$. Since l is injective we have $l(x) \neq l(y)$ and hence $l(x) \perp l(y)$. Suppose that $p' < q'$. This means that for all $e' \in \mathscr{E}'$ we have $O'(e', p') \subset O'(e', q')$. Consider an arbitrary $e \in \mathscr{E}$; then $l(O(e, m(p'))) = O'(n(e), p') \subset O'(n(e), q') = l(O(e, m(q')))$. Since l is injective this shows that $O(e, m(p')) \subset O(e, m(q'))$. This proves that $p < q$. Suppose that $e \perp f$; this means that there exists $p \in \Sigma$ such that $O(e, p) \cap O(f, p) = \emptyset$. This implies that $l(O(e, p) \cap O(f, p)) = l(O(e, p)) \cap l(O(f, p)) = \emptyset$. Consider $p' \in \Sigma'$ such that $m(p') = p$, then $O'(n(e), p') \cap O'(n(f), p') = l(O(e, m(p'))) \cap l(O(f, m(p'))) = \emptyset$. This shows that $n(e) \perp n(f)$. We have $(e, m(p')) < (f, m(q')) \Leftrightarrow O(e, m(p')) \subset O(f, m(q')) \Leftrightarrow l(O(e, m(p'))) \subset l(O(f, m(q'))) \Leftrightarrow O'(n(e), p') \subset O'(n(f), q')) \Leftrightarrow (n(e), p') < (n(f), q')$. We have $(e, m(p')) \perp (f, m(q')) \Leftrightarrow O(e, m(p')) \cap O(f, m(q')) = \emptyset \Leftrightarrow l(O(e, m(p'))) \cap l(O(f, m(q'))) = \emptyset \Leftrightarrow O'(n(e), p') \cap O'(n(f), q')) = \emptyset \Leftrightarrow (n(e), p') \perp (n(f), q')$.

We remark that the function m does not necessarily conserve the orthogonality relation. It can be that states of S' that are orthogonal are mapped onto states of S that are not orthogonal. In the same way, the function n does not necessarily conserve the preorder relation. It can well be that experiments that 'imply' each other for the subentity do not 'imply' each other for the entity. But both functions are 'continuous' for the eigenclosure system.

Proposition 28. Suppose that $S(\mathscr{E}, \Sigma, X, \mathbb{O})$ and $S'(\mathscr{E}', \Sigma', X', \mathbb{O}')$ are two entities, such that S is a subentity of S', and m, n, and l are the connecting functions related to S and S'. For $p' \in \Sigma'$, $e \in \mathscr{E}$, $A \subset X$, and $A' \subset X'$ we have

$$m^{-1}(eig_e(A)) = eig_{n(e)}(l(A)) \tag{135}$$

$$n^{-1}(eig'_{p'}(A')) = eig_{m(p')}(l^{-1}\ (A')) \tag{136}$$

Proof. We have $p' \in m^{-1}(eig_e(A)) \Leftrightarrow m(p') \in eig_e(A) \Leftrightarrow O(e, m(p')) \subset A \Leftrightarrow l(O(e, m(p')) \subset l(A) \Leftrightarrow O(n(e), p') \subset l(A) \Leftrightarrow p' \in eig_{n(e)}(l(A))$. We also have $e \in n^{-1}(eig'_{p'}(A')) \Leftrightarrow n(e) \in eig'_{p'}(A') \Leftrightarrow O'(n(e), p') \subset A' \Leftrightarrow l(O(e,m(p')) \subset A' \Leftrightarrow O(e,m(p')) \subset l^{-1}(A') \Leftrightarrow e \in eig_{m(p')}(l^{-1}(A'))$.

Theorem 41. Suppose that $S(\mathscr{E}, \Sigma, X, \mathbb{O})$ and $S'(\mathscr{E}', \Sigma', X', \mathbb{O}')$ are two entities, such that S is a subentity of S', and m, n, and l are the connecting functions related to S and S'. Then m and n are continuous functions for the eigenclosure systems, or

$$F \in \mathscr{F}_{eig} \Rightarrow m^{-1}\ (F) \in \mathscr{F}'_{eig} \tag{137}$$

$$G' \in \mathscr{G}'_{eig} \Rightarrow n^{-1}(G) \in \mathscr{G}_{eig} \tag{138}$$

Proof. Suppose that $F \in \mathscr{F}_{eig}$. Then we have $F = \cap_{e \in \mathscr{E}} F_e$ with $F_e \in \mathscr{F}_{eig}(e)$. From the foregoing theorem it follows that for each $F_e \in \mathscr{F}_{eig}(e)$ we have $m^{-1}(F_e) \in \mathscr{F}'_{eig}(n(e))$ and hence $m^{-1}(F_e) \in \mathscr{F}'_{eig}$. We have $m^{-1}(\cap_{e \in \mathscr{E}} F_e) = \cap_{e \in \mathscr{E}} m^{-1}(F_e)$, which shows that $m^{-1}(F) \in \mathscr{F}'_{eig}$. Suppose that $G' \in \mathscr{G}'_{eig}$; then we have that $G' = \cap_{p' \in \Sigma'} G_{p'}$, where $G_{p'} \in \mathscr{G}'_{eig}(p')$. Hence we have $n^{-1}(G_{p'}) \in \mathscr{G}_{eig}(m(p'))$ and hence also $n^{-1}(G_{p'}) \in \mathscr{G}_{eig}$. Since we have $n^{-1}(\cap_{p' \in \Sigma'} G_{p'}) = \cap_{p' \in \Sigma'} n^{-1}(G_{p'})$ we have $n^{-1}(G) \in \mathscr{G}_{eig}$.

Let us consider the situation of two probabilistic entities S and S' such that S is a subentity of S' and let $\mathcal{M}$ be the set of generalized probability measures of S and $\mathcal{M}'$ the set of generalized probability measures of S'. We will call S a 'probabilistic subentity' of S' if the respective generalized probability measures are connected in the way we will specify now. We recall that the situation that we consider is the following: if we perform an experiment e on the entity S in a state $m(p')$ where p' is a state of S', then the outcome $x(e, m(p'))$ occurs iff one specific outcome $x'(n(e), p') = l(x(e, m(p')))$ occurs after the performance of the experiment $n(e)$ on the entity S' in state p'. This means that if we perform repeated experiments on entities in identical states and calculate the relative frequencies of outcomes $x(e, m(p'))$ and outcomes $x'(n(e), p')$, they will be the same. This means that also the limits of these relative frequencies, i.e., the probabilities, will match.

Suppose that $S(\mathscr{E}, \Sigma, X, \mathbb{O}, \mathscr{M})$ and $S'(\mathscr{E}', \Sigma', X', \mathbb{O}', \mathscr{M}')$ are two probabilistic entities. To each $\mu \in \mathscr{M}$ corresponds a $\mu' \in \mathscr{M}'$ such that μ represents the relative frequency operation on the subentity S and μ' represents the corresponding relative frequency operation on the entity S'. And we have $\mu(e, m(p'), x) = \mu'(n(e), p', l(x))$. Let us formalize this physical idea.

Definition 48. Suppose that $S(\mathscr{E}, \Sigma, X, \mathbb{O}, \mathscr{M})$ and $S'(\mathscr{E}', \Sigma', X', \mathbb{O}', \mathscr{M}')$ are two probabilistic entities such that S is a subentity of S'. We will say that S is a probabilistic subentity iff there exists an injective function

$$k: \quad \mathscr{M} \rightarrow \mathscr{M}', \quad \mu \mapsto k(\mu) \tag{139}$$

such that for $e \in \mathscr{E}$, $p' \in \Sigma'$, and $x \in X$ we have

$$\mu(e, m(p'), x) = k(\mu)(n(e), p', l(x)) \tag{140}$$

14. A FINITE EXAMPLE

The first example that we discuss is a finite example. Let us consider an entity S with the following set of states Σ, set of experiments $\mathscr{E}$, and sets of outcomes:

$$\Sigma = \{p, q, r\}, \qquad \mathscr{E} = \{e, f, g\} \tag{141}$$

$$\begin{array}{lll} O(e,p) = \{x_1, x_2\}, & O(e, q) = \{x_1, x_3\}, & O(e,r) = \{x_2, x_3\} \\ O(f, p) = \{y_1, y_2\}, & O(f, q) = \{x_2, y_2\}, & O(f, r) = \{x_3, y_1, y_2\} \\ O(g, p) = \{x_1, y_1\}, & O(g, q) = \{x_2\}, & O(g, r) = \{x_1, x_2, y_1\} \end{array} \tag{142}$$

Then we have

$$\begin{array}{lll} O(e) = \{x_1, x_2, x_3\}, & O(f) = \{x_2, x_3, y_1, y_2\}, & O(g) = \{x_1, x_2, y_1\} \\ O(p) = \{x_1, x_2, y_1, y_2\}, & O(q) = \{x_1, x_2, x_3\} & O(r) = \{x_1, x_2, x_3, y_1, y_2\} \end{array}$$

$$X = \{x_1, x_2, x_3, y_1, y_2\}$$

14.1. Preorder and Orthogonality

We introduce a shorter notation for the nine elements of $\mathscr{E} \times \Sigma$. Let us denote $(e, p) = \lambda_{11}$, $(e, q) = \lambda_{12}$, $(e, r) = \lambda_{13}$, $(f, p) = \lambda_{21}$, $(f, q) = \lambda_{22}$, $(f, r) = \lambda_{23}$, $(g, p) = \lambda_{31}$, $(g, q) = \lambda_{32}$, and $(g, r) = \lambda_{33}$. We have

$$\begin{array}{lllll} \lambda_{11} < \lambda_{33} & \lambda_{21} < \lambda_{23} & \lambda_{32} < \lambda_{11} & \lambda_{32} < \lambda_{13} & \lambda_{32} < \lambda_{22} \\ \lambda_{32} < \lambda_{33} & \lambda_{11} \perp \lambda_{21} & \lambda_{11} \perp \lambda_{23} & \lambda_{12} \perp \lambda_{21} & \lambda_{12} \perp \lambda_{22} \\ \lambda_{12} \perp \lambda_{32} & \lambda_{13} \perp \lambda_{21} & \lambda_{13} \perp \lambda_{31} & \lambda_{21} \perp \lambda_{11} & \lambda_{21} \perp \lambda_{12} \\ \lambda_{21} \perp \lambda_{13} & \lambda_{21} \perp \lambda_{32} & \lambda_{22} \perp \lambda_{12} & \lambda_{22} \perp \lambda_{31} & \lambda_{23} \perp \lambda_{11} \\ \lambda_{23} \perp \lambda_{32} & \lambda_{31} \perp \lambda_{13} & \lambda_{31} \perp \lambda_{22} & \lambda_{31} \perp \lambda_{32} & \lambda_{32} \perp \lambda_{12} \\ \lambda_{32} \perp \lambda_{21} & \lambda_{32} \perp \lambda_{23} & \lambda_{32} \perp \lambda_{31} & & \end{array} \tag{144}$$

Let us now calculate the preorder and orthogonality relations on the set of states Σ and on the set of experiments $\mathscr{E}$ for this example. We have

$$\begin{array}{lllllll} p \not< q & q \not< p & p \not< r & r \not< p & q \not< r & r \not< q & p \perp_g q \\ p \not\perp r & q \not\perp r & e \not< f & f \not< e & e \not< g & g \not< e & f \not< g \\ q \not< f & e \perp_p f & e \perp_q f & e \perp_q g & f \not\perp g & & \end{array} \tag{145}$$

We have q that is an eigenstate of g with eigenoutcome x_2, and hence (g, q) is an eigencouple with eigenoutcome x_2.

14.2. The Eigenclosures

Let us now study the closure structures and let us construct the eigenmap eig and the closure system $\mathscr{Y}$ for our finite example. We have

$$\begin{array}{rr} eig(\{x_2\}) = \{\lambda_{32}\}, & eig(\{x_1, x_2\}) = \{\lambda_{11}, \lambda_{32}\} \\ eig(\{x_2, x_3\}) = \{\lambda_{13}, \lambda_{32}\}, & eig(\{x_1, y_1\}) = \{\lambda_{31}\} \\ eig(\{x_1, x_3\}) = \{\lambda_{12}\}, & eig(\{x_1, x_3, y_2\}) = \{\lambda_{12}\} \\ eig(\{x_2, y_1\}) = \{\lambda_{32}\}, & eig(\{x_2, y_2\}) = \{\lambda_{22}, \lambda_{32}\} \\ eig(\{y_1, y_2\}) = \{\lambda_{21}\}, & eig(\{x_1, x_2, x_3\}) = \{\lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{32}\} \\ eig(\{x_1, x_3, y_1\}) = \{\lambda_{12}, \lambda_{31}\}, & eig(\{x_1, x_2, y_1\}) = \{\lambda_{11}, \lambda_{31}, \lambda_{32}, \lambda_{33}\} \\ eig(\{x_1, y_1, y_2\}) = \{\lambda_{21}, \lambda_{31}\}, & eig(\{x_1, x_2, y_2\}) = \{\lambda_{11}, \lambda_{22}, \lambda_{32}\} \\ eig(\{x_2, y_3, y_1\}) = \{\lambda_{13}, \lambda_{32}\}, & eig(\{x_2, x_3, y_2\}) = \{\lambda_{13}, \lambda_{32}, \lambda_{22}\} \\ eig(\{x_3, y_1, y_2\}) = \{\lambda_{21}, \lambda_{23}\}, & eig(\{x_2, y_1, y_2\}) = \{\lambda_{21}, \lambda_{22}, \lambda_{32}\} \end{array} \tag{146}$$

$$eig(\{x_1, x_3, y_1, y_2\}) = \{\lambda_{12}, \lambda_{21}, \lambda_{23}, \lambda_{31}\}$$

$$eig(\{x_1, x_2, x_3, y_2\}) = \{\lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{22}, \lambda_{32}\}$$

$$eig(\{x_2, x_3, y_1, y_2\}) = \{\lambda_{13}, \lambda_{21}, \lambda_{22}, \lambda_{23}, \lambda_{32}\}$$

$$eig(\{x_1, x_2, y_1, y_2\}) = \{\lambda_{11}, \lambda_{21}, \lambda_{22}, \lambda_{31}, \lambda_{32}, \lambda_{33}\}$$

$$eig(\{x_1, x_2, x_3, y_1\}) = \{\lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{31}, \lambda_{32}, \lambda_{33}\}$$

The images of all other subsets of X are $\emptyset$ or $\mathscr{E} \times \Sigma$ or already contained in the ones presented here. Hence, if $\mathscr{Y}_{eig}$ is the set of eigenclosed subsets, we have

$$\begin{aligned}\mathscr{Y}_{eig} = \{&\emptyset, \{\lambda_{12}\}, \{\lambda_{31}\}, \{\lambda_{32}\}, \{\lambda_{21}\}, \{\lambda_{11}, \lambda_{32}\}, \{\lambda_{13}, \lambda_{32}\}, \{\lambda_{21}, \lambda_{31}\},\\ &\{\lambda_{32}, \lambda_{22}\}, \{\lambda_{22}, \lambda_{21}, \lambda_{32}\}, \{\lambda_{21}, \lambda_{23}\}, \{\lambda_{11}, \lambda_{31}, \lambda_{32}, \lambda_{33}\},\\ &\{\lambda_{11}, \lambda_{22}, \lambda_{32}\}, \{\lambda_{12}, \lambda_{31}\}, \{\lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{32}\}, \{\lambda_{12}, \lambda_{21}, \lambda_{23}, \lambda_{31}\},\\ &\{\lambda_{13}, \lambda_{32}, \lambda_{22}\}, \{\lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{22}\}, \{\lambda_{13}, \lambda_{21}, \lambda_{22}, \lambda_{23}, \lambda_{32}\},\\ &\{\lambda_{11}, \lambda_{21}, \lambda_{22}, \lambda_{31}, \lambda_{32}, \lambda_{33}\}, \{\lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{31}, \lambda_{32}, \lambda_{33}\}\}\end{aligned} \tag{147}$$

Let us now construct the closure system on the set of states. We have

$$eig_e(\{x_1, x_2\}) = \{p\}, \qquad eig_e(\{x_1, x_3\}) = \{q\}, \qquad eig_e(\{x_2, x_3\}) = \{r\} \tag{148}$$

and all the other images of eig_e are $\emptyset$ or Σ. This shows that

$$\mathscr{F}(e) = \{\emptyset, \{p\}, \{q\}, \{r\}, \Sigma\} \tag{149}$$

We also have

$$\begin{aligned} eig_f(\{y_1, y_2\}) = \{p\} \qquad & eig_f(\{y_2, x_2\}) = \{q\} \\ eig_f(\{y_1, y_2, x_3\}) = \{p, r\}, \qquad & eig_f(\{y_1, y_2, x_2\}) = \{p, q\} \end{aligned} \tag{150}$$

and the other images that are $\emptyset$ or Σ. This shows that

$$\mathscr{F}(f) = \{\emptyset, \{p\}, \{q\}, \{p, q\}, \{p, r\}, \Sigma\} \tag{151}$$

And finally we have

$$eig_g(\{x_2\}) = \{q\}, \qquad eig_g(\{x_1, y_1\}) = \{p\} \tag{152}$$

which shows that

$$\mathscr{F}(g) = \{\emptyset, \{p\}, \{q\}, \Sigma\} \tag{153}$$

The state eigenclosure system is given by

$$\mathcal{F} = \mathcal{A} = \{\emptyset, \{p\}, \{q\}, \{r\}, \{p, q\}, \{p, r\}, \Sigma\} \tag{154}$$

Let us now construct the closure system on the set of experiments. We have

$$\begin{aligned} &eig_p(\{x_1, x_2\}) = \{e\}, \qquad && eig_p(\{y_1, y_2\}) = \{f\} \\ &eig_p(\{x_1, x_2, y_1\}) = \{e, g\}, \qquad && eig_p(\{x_1, y_1, y_2\}) = \{f, g\} \end{aligned} \tag{155}$$

We have then

$$\mathcal{G}(p) = \{\emptyset, \{e\}, \{f\}, \{g\}, \{e, g\}, \{f, g\}, \mathcal{E}\} \tag{156}$$

For the state q we have

$$\begin{aligned} &eig_q(\{x_1, x_3\}) = \{e\}, \qquad && eig_q(\{x_2\}) = \{g\} \\ &eig_q(\{x_2, y_2\}) = \{f, g\}, \qquad && eig_q(\{x_1, x_2, x_3\}) = \{e, g\} \end{aligned} \tag{157}$$

We have then

$$\mathcal{G}(q) = \{\emptyset, \{e\}, \{g\}, \{e, g\}, \{f, g\}, \mathcal{E}\} \tag{158}$$

Finally for r we have

$$\begin{aligned} &eig_r(\{x_2, x_3\}) = \{e\}, \qquad && eig_r(\{y_1, y_2, x_3\}) = \{f\} \\ &eig_r(\{x_1, x_2, y_1\}) = \{g\}, \qquad && eig_r(\{y_1, y_2, x_2, x_3\}) = \{e, f\} \\ &eig_r(\{x_1, x_2, x_3, y_1\}) = \{e, g\} \end{aligned} \tag{159}$$

We have then

$$\mathcal{G}(r) = \{\emptyset, \{e\}, \{f\}, \{g\}, \{e, g\}, \{e, f\}, \mathcal{E}\} \tag{160}$$

This means that

$$\mathcal{G} = \{\emptyset, \{e\}, \{f\}, \{g\}, \{e, f\}, \{e, g\}, \{f, g\}, \mathcal{E}\} = \mathcal{P}(\mathcal{E}) \tag{161}$$

We can easily see that in general $\mathcal{Y}$(state) is different from $\mathcal{F}$ by considering our example. Indeed we have

$$eig(\{x_1, x_2, y_1, y_2\})(\text{state}) = \{p\} \tag{162}$$

All the other traces from elements of $\mathcal{Y}$ are $\emptyset$ or Σ, which shows that

$$\mathcal{Y}(\text{state}) = \{\emptyset, \{p\}, \Sigma\} \tag{163}$$

This shows that, for example, $\{q\}$ is not contained in $\mathcal{Y}$(state), while it is contained in $\mathcal{F}$.

14.3. The Orthoclosures

Let us construct now the orthoclosure systems $\mathcal{Y}_{orth}$, $\mathcal{F}_{orth}(e)$, and $\mathcal{F}_{orth}$. To do this, we first construct the generating set of elements consisting of the orthogonals of singletons. First we construct; $\mathcal{Y}_{orth}$.

$$\begin{aligned}
&\{\lambda_{11}\}^{\perp} = \{\lambda_{21}, \lambda_{23}\}, &&\{\lambda_{12}\}^{\perp} = \{\lambda_{21}, \lambda_{22}, \lambda_{32}\} \\
&\{\lambda_{13}\}^{\perp} = \{\lambda_{21}, \lambda_{31}\}, &&\{\lambda_{21}\}^{\perp} = \{\lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{32}\} \\
&\{\lambda_{22}\}^{\perp} = \{\lambda_{12}, \lambda_{31}\}, &&\{\lambda_{23}\}^{\perp} = \{\lambda_{11}, \lambda_{32}\} \\
&\{\lambda_{31}\}^{\perp} = \{\lambda_{13}, \lambda_{22}, \lambda_{32}\}, &&\{\lambda_{32}\}^{\perp} = \{\lambda_{12}, \lambda_{21}, \lambda_{23}, \lambda_{31}\} \\
&\{\lambda_{33}\}^{\perp} = \emptyset,
\end{aligned} \tag{164}$$

If we consider this collection as a generating set of elements we find

$$\begin{aligned}
\mathcal{Y}_{orth} = \{&\emptyset, \{\lambda_{12}\}, \{\lambda_{21}\}, \{\lambda_{31}\}, \{\lambda_{32}\}, \{\lambda_{11}, \lambda_{32}\}, \{\lambda_{13}, \lambda_{32}\}, \{\lambda_{22}, \lambda_{32}\}, \\
&\{\lambda_{23}, \lambda_{21}\}, \{\lambda_{21}, \lambda_{31}\}, \{\lambda_{12}, \lambda_{31}\}, \{\lambda_{21}, \lambda_{22}, \lambda_{32}\}, \{\lambda_{13}, \lambda_{22}, \lambda_{32}\}, \\
&\{\lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{32}\}, \{\lambda_{12}, \lambda_{21}, \lambda_{23}, \lambda_{31}\}, \mathcal{E} \times \Sigma\}
\end{aligned} \tag{165}$$

Let us now construct $\mathcal{F}_{orth}(e)$, $\mathcal{F}_{orth}(f)$, and $\mathcal{F}_{orth}(g)$. We have

$$\{p\}^{\perp_e} = \emptyset, \qquad \{q\}^{\perp_e} = \emptyset, \qquad \{r\}^{\perp_e} = \emptyset \tag{166}$$

From this it follows that

$$\mathcal{F}_{orth}(e) = \{\emptyset, \Sigma\} \tag{167}$$

In an analogous way we have

$$\mathcal{F}_{orth}(f) = \{\emptyset, \Sigma\} \tag{168}$$

Let us now construct $\mathcal{F}_{orth}(g)$. We have

$$\{p\}^{\perp_g} = \{q\}, \qquad \{q\}^{\perp_g} = \{p\}, \qquad \{r\}^{\perp_g} = \emptyset \tag{169}$$

From this it follows that

$$\mathcal{F}_{orth}(g) = \{\emptyset, \{p\}, \{q\}, \Sigma\} \tag{170}$$

It also follows that

$$\mathcal{F}_{orth} = \{\emptyset, \{p\}, \{q\}, \Sigma\} \tag{171}$$

Again we can see that the trace of the orthoclosure system is not equal to the state orthoclosure system in general. Indeed we have

$$\mathcal{Y}_{orth}(\text{state}) = \{\emptyset, \{p\}, \Sigma\} \tag{172}$$

The example shows us that the eigenclosures are in general different from the orthoclosures.

14.4. Special Properties

We can easily check that our entity is 'outcome determined.' Let us calculate the eigenclosures of the singletons. We have

$$\begin{aligned}
cl_{eig}(\{\lambda_{11}\}) &= \{\lambda_{11}, \lambda_{32}\} = eig(O(\lambda_{11})) \\
cl_{eig}(\{\lambda_{12}\}) &= \{\lambda_{12}\} = eig(O(\lambda_{12})) \\
cl_{eig}(\{\lambda_{13}\}) &= \{\lambda_{13}, \lambda_{32}\} = eig(O(\lambda_{13})) \\
cl_{eig}(\{\lambda_{21}\}) &= \{\lambda_{21}\} = eig(O(\lambda_{21})) \\
cl_{eig}(\{\lambda_{22}\}) &= \{\lambda_{22}, \lambda_{32}\} = eig(O(\lambda_{22})) \\
cl_{eig}(\{\lambda_{23}\}) &= \{\lambda_{22}, \lambda_{23}\} = eig(O(\lambda_{23})) \\
cl_{eig}(\{\lambda_{31}\}) &= \{\lambda_{31}\} = eig(O(\lambda_{31})) \\
cl_{eig}(\{\lambda_{32}\}) &= \{\lambda_{32}\} = eig(O(\lambda_{32})) \\
cl_{eig}(\{\lambda_{33}\}) &= \{\lambda_{11}, \lambda_{31}, \lambda_{32}, \lambda_{33}\} = eig(O(\lambda_{33}))
\end{aligned} \tag{173}$$

In this example we can also see that the orthoclosure of the singletons is not necessarily equal to the eigenclosure, even in the case of an 'outcome determined' entity. Indeed, for example,

$$cl_{orth}(\{\lambda_{33}\}) = \mathcal{E} \times \Sigma \neq cl_{eig}(\{\lambda_{33}\}) \tag{174}$$

15. STANDARD QUANTUM MECHANICS

We describe now the way in which our formalism is related to the complex Hilbert space model of standard quantum mechanics. We will introduce the concepts of our approach and illustrate what they are for standard quantum mechanics. We will see that everything works very well except when we arrive at the description of subentities. There something peculiar happens, as was remarked early in quantum mechanics, and has been studied in detail in Aerts and Daubechies (1978) and Aerts (1981, 1982, 1984a). We will come back to the problem of the description of subentities in the next section and a proposal for its solution will lead us to the formulation of an alternative quantum mechanics in Hilbert space where additional 'pure' states are introduced in a very natural way. Let us first describe the nonproblematic aspects of standard quantum mechanics.

For simplicity of notation we consider a finite dimensional complex Hilbert space, but it is easy to see that an analogous scheme can be formulated for the case of a separable infinite-dimensional complex Hilbert space. Hence consider the n-dimensional complex Hilbert space $\mathcal{H}$. Let us first introduce some concepts of the Hilbert space that we will use in the following.

Definition 49. Consider a separable complex Hilbert space $\mathcal{H}$. We introduce the set of unit vectors, the set of rays, the set of orthogonal projections, and the set of spectral families of the Hilbert space:

$$\begin{aligned} \mathcal{U}(\mathcal{H}) &= \{c \mid c \in \mathcal{H}, \|c\| = 1\} \\ \mathcal{R}(\mathcal{H}) &= \{\bar{c} \mid \bar{c} \text{ is the ray of } \mathcal{H} \text{ generated by } c \in \mathcal{U}(\mathcal{H})\} \\ \mathcal{P}(\mathcal{H}) &= \{E_k \mid E_k \text{ is an orthogonal projection of } \mathcal{H}\} \\ \mathcal{S}(\mathcal{H}) &= \{E \mid E \text{ is a spectral family of } \mathcal{H}\} \end{aligned} \tag{175}$$

We will denote unit vectors by c, d, . . . , rays by $\bar{c}$, $\bar{d}$, . . . , orthogonal projections by E_k, E_l, . . . , and spectral families by E, D, . . . ,

For an entity that is described by this Hilbert space in standard quantum theory a state $p_{\bar{c}}$ is represented by a ray $\bar{c} \in \mathcal{R}(\mathcal{H})$ of the Hilbert space (this will no longer be the case in the alternative completed quantum mechanics that we present in the next section).

Traditionally it is said that an experiment is described by a self-adjoint operator. However, if we want to remain closer to the physical meaning, it is well known that we can better represent the experiment by means of the spectral family of orthogonal projections of this self-adjoint operator. Let us first mention the spectral theorem that makes both representations equivalent.

Proposition 29. If H is a self-adjoint operator of an n-dimensional complex Hilbert space $\mathcal{H}$, then there exist distinct real numbers $\lambda_1, \ldots, \lambda_r$ $(1 \leq r \leq n)$ and a pairwise orthogonal set of nonzero projections $\{E_1, \ldots, E_r\}$ such that

$$\sum_{k=1}^{r} E_k = 1, \qquad H = \sum_{k=1}^{r} \lambda_k E_k \tag{176}$$

which will be called a 'spectral family' of the Hilbert space $\mathcal{H}$. Conversely, if $\{\lambda_1, \ldots, \lambda_r\}$ is a set of distinct real numbers and $\{E_1, \ldots, E_r\}$ is a pairwise orthogonal set of nonzero projections, and if the above two conditions are satisfied and hence we have a spectral family, then $\{\lambda_1, \ldots, \lambda_r\}$ is the set of distinct eigenvalues of H, and for each k, E_k is the projection onto the eigenspace corresponding to λ_k.

That is the reason that we shall represent an experiment by the spectral family $E = \{E_1, \ldots, E_r\}$ of pairwise orthogonal nonzero projections that satisfies the first of the two conditions mentioned in the spectral theorem. We will not use the λ_i to indicate the outcomes, although we could do so, but it would show less the underlying structure of the outcomes. Instead we identify an outcome x_{E_k} in the quantum model with the eigenspace E_k of the Hilbert space (or with the orthogonal projector E_k on this eigenspace; we will not make a distinction). The set of all outcomes X_{sq} for the standard quantum model corresponds to the set of all orthogonal projections or equivalently the set of all closed subspaces of the Hilbert space $\mathcal{P}(\mathcal{H})$, which is a complete atomic orthocomplemented lattice. For an experiment e_E we have $O(e_E) = \{x_{E_1}, \ldots, x_{E_r}\}$. Suppose that the entity is in state $p_{\bar{c}}$ and we consider an experiment e_E; then the set of outcomes $O(e_E, p_{\bar{c}})$ is determined in the following way: for $x_{E_j} \in O(e_E)$ we have $x_{E_j} \in O(e_E, p_{\bar{c}}) \Leftrightarrow E_j(c) \neq 0$.

Let us now identify the probabilities as they appear in the case of a quantum entity described by the standard quantum mechanical formalism. A quantum entity is a probabilistic entity where the probabilities are defined as follows. Suppose that we have an experiment e_E, a state $p_{\bar{c}}$, and an outcome $x_{E_k} \in O(e_E, p_{\bar{c}})$; then $\mu(e_E, p_{\bar{c}}, x_{E_k}) = \langle c, E_k(c)\rangle$, where $\langle . , . \rangle$ is the inproduct of the Hilbert space, is the probability that the outcome x_{E_k} occurs if the experiment e_E is performed, the entity being in state $p_{\bar{c}}$. It is interesting to remark that the quantum probabilities only depend on the state and the outcome and not on the experiment. This is one of the essential features of standard quantum mechanics. We have now introduced all the necessary correspondences to present a formal definition.

Definition 50. Consider a probabilistic entity $S(\mathcal{E}_{sq}, \Sigma_{sq}, X_{sq}, \mathbb{O}_{sq}, \mathcal{M}_{sq})$ and a separable complex Hilbert space $\mathcal{H}$, with set of unit vectors $\mathcal{U}(\mathcal{H})$, a set of rays $\mathcal{R}(\mathcal{H})$, a set of orthogonal projections $\mathcal{P}(\mathcal{H})$, and a set of spectral families $\mathcal{S}(\mathcal{H})$. We say that the entity is a 'standard quantum entity' iff we have

$$\begin{aligned} \mathcal{E}_{sq} &= \{e_E | E \in \mathcal{S}(\mathcal{H})\} \\ \Sigma_{sq} &= \{p_{\bar{c}} | \bar{c} \in \mathcal{R}(\mathcal{H})\} \\ X_{sq} &= \{x_{E_k} | E_k \in \mathcal{P}(\mathcal{H})\} \\ \mathbb{O}_{sq} &= \{O(e_E, p_{\bar{c}}) | E \in \mathcal{S}(\mathcal{H}), \bar{c} \in \mathcal{R}(\mathcal{H})\} \\ \mathcal{M}_{sq} &= \{\mu | \mu: \mathcal{E}_{sq} \times \Sigma_{sq} \times X_{sq} \to [0, 1] \text{ is a generalized probability}\} \end{aligned} \tag{177}$$

such that

$$O(e_E, p_{\bar{c}}) = \{x_{E_k} | E_k \in \mathcal{P}(\mathcal{H}),\ E_k(c) \neq 0\}$$

$$\mu(e_E, p_{\bar{c}}, E_k) = \langle c, E_k c \rangle \qquad \text{if} \quad E_k \in E \tag{178}$$

$$\mu(e_E, p_{\bar{c}}, E_k) = 0 \qquad \text{if} \quad E_k \notin E$$

15.1. Preorder and Orthogonality

Let us investigate the orthogonality relation and show that it coincides with the orthogonality of the Hilbert space.

Proposition 30. Consider a standard quantum entity $S(\mathscr{E}_{sq}, \Sigma_{sq}, X_{sq}, \mathbb{O}_{sq}, \mathcal{M}_{sq})$. If $x_{E_1}, x_{E_2} \in X_{sq}$ then

$$x_{E_1} \perp x_{E_2} \Leftrightarrow E_1 \perp E_2 \tag{179}$$

Proof. Suppose that $x_{E_1} \perp x_{E_2}$; then there exists $e_E \in \mathscr{E}_{sq}$ and $p_{\bar{c}} \in \Sigma_{sq}$ such that $x_{E_1} \neq x_{E_2} \in O(e_E, p_{\bar{c}})$. By definition of e_E it follows that $E_1, E_2 \in E$ and hence $E_1 \perp E_2$. If, on the other hand, $E_1 \perp E_2$, it is always possible to consider a spectral family E such that $E_1, E_2 \in E$. Further, we can choose easily a vector c such that $E_1(c) \neq 0$ and $E_2(c) \neq 0$. Then we have that $x_{E_1}, x_{E_2} \in O\ (e_E, p_{\bar{c}})$, which proves that $x_{E_1} \perp x_{E_2}$.

It is important to show that the orthogonality relation on the set of states coincides with the original orthogonality relation in the Hilbert space.

Proposition 31. Consider a standard quantum entity $S(\mathscr{E}_{sq}, \Sigma_{sq}, \mathscr{X}_{sq}, \mathbb{O}_{sq}, \mathcal{M}_{sq})$. For $p_{\bar{c}}, p_{\bar{d}} \in \Sigma_{sq}$, we have

$$p_{\bar{c}} \perp p_{\bar{d}} \Leftrightarrow c \perp d \tag{180}$$

Proof. Suppose that $p_{\bar{c}} \perp p_{\bar{d}}$; then there exists an experiment e_E, with $E = \{E_1, \ldots, E_r\}$, such that $O(e_E, p_{\bar{c}}) \cap O(e_E, p_{\bar{d}}) = \emptyset$. This means that we have two subsets $K \subset \{1, \ldots, r\}$ and $L \subset \{1, \ldots, r\}$ such that $K \cap L = \emptyset$ and $O(e_E, p_{\bar{c}}) = \{x_{E_i} | i \in K\}$, while $O(e_E, p_{\bar{d}}) = \{x_{E_i} | i \in L\}$. We have $E_i(c) \neq 0$ for $i \in K$ and $E_i(d) \neq 0$ for $i \in L$. This implies that $E_i(c) = 0$ for $i \notin K$ and $E_i(d) = 0$ for $i \notin L$, which shows that $\Sigma_{i \notin K} E_i(c) = 0$ and $\Sigma_{i \notin L} E_i(d) = 0$. And since E_i, $i \in \{1, \ldots, r\}$, is a spectral family we have $\Sigma_{i \in K} E_i(c) = c$ and $\Sigma_{i \in L} E_i(d) = d$, which shows that $c \perp d$. The other implication is straightforward.

Proposition 32. Consider a standard quantum entity $S(\mathscr{E}_{sq}, \Sigma_{sq}, X_{sq}, \mathbb{O}_{sq}, \mathcal{M}_{sq})$. For $p_{\bar{c}}, p_{\bar{d}} \in \Sigma_{sq}$ we have

$$p_{\bar{c}} < p_{\bar{d}} \Leftrightarrow \bar{c} = \bar{d} \Leftrightarrow p_{\bar{c}} = p_d \tag{181}$$

Proof. Suppose that $\bar{c} \neq \bar{d}$. We do not have to consider the situation where $\bar{c} \perp \bar{d}$ since then certainly $p_{\bar{c}} \not< p_{\bar{d}}$. Hence suppose that $\bar{c} \not\perp \bar{d}$. Let us construct an experiment by means of a set of spectral projections $\{E_1, \ldots, E_r\}$ where E_k is a one-dimensional projector that is orthogonal to $\bar{d}$, but not orthogonal to $\bar{c}$. This is always possible if the Hilbert space has dimension ≥ 2. For this experiment e_E we have that $O(e_E, p_{\bar{c}})$ contains the outcome x_{E_k}, while $O(e_E, p_{\bar{d}})$ does not contain it. This shows that $p_{\bar{c}} \not< p_{\bar{d}}$. If the Hilbert space has dimension 1, the proposition is trivially satisfied.

Theorem 42. A standard quantum entity $S(\mathscr{E}_{sq}, \Sigma_{sq}, X_{sq}, \mathbb{O}_{sq}, \mathcal{M}_{sq})$ is state atomic.

Proposition 33. Consider a standard quantum entity $S(\mathscr{E}_{sq}, \Sigma_{sq}, X_{sq}, \mathbb{O}_{sq}, \mathcal{M}_{sq})$. For $e_E, e_F \in \mathscr{E}_{sq}$ we have

$$e_E = e_F \qquad \text{or} \qquad e_E \perp e_F \tag{182}$$

Proof. For a Hilbert space of dimension 1 the proposition is trivially satisfied. Hence consider a Hilbert space of at least dimension 2. Consider two experiments $e_E \neq e_F$. This situation is of the following nature. We have $E = \{E_1, \ldots, E_s, E_{s+1}, \ldots, E_r\}$ and $F = \{E_1, \ldots, E_s, F_{s+1}, \ldots, F_i\}$, where s is the number of spectral projections that are equal; hence $F_i \neq E_j$. Let us take now a vector $c \in (\Sigma_{i=1}^{s} E_i)^{\perp}$, which is always possible since $e_E \neq e_F$, i.e., $E \neq F$. We then have $O(e_E, p_{\bar{c}}) \cap O(e_F, p_{\bar{c}}) = \emptyset$, which proves that $e_E \perp e_F$.

For the orthogonality and preorder relation on $\mathscr{E} \times \Sigma$ different situations are possible.

Proposition 34. Consider a standard quantum entity $S(\mathscr{E}_{sq}, \Sigma_{sq}, X_{sq}, \mathbb{O}_{sq}, \mathcal{M}_{sq})$. For $(e_E, p_{\bar{c}}), (e_E, p_{\bar{d}}) \in \mathscr{E}_{sq} \times \Sigma_{sq}$ we have

$$(e_E, p_{\bar{c}}) < (e_E, p_{\bar{d}}) \Leftrightarrow R(c) = c \tag{183}$$

where

$$R = \sum_{x_{E_k} \in O(e_E, p_{\bar{d}})} E_k \tag{184}$$

Proof. We have $R(c) = c \Leftrightarrow E_k(c) = 0$ for $x_{E_k} \notin O(e_E, p_{\bar{d}}) \Leftrightarrow O(e_E, p_{\bar{c}}) \subset O(e_E, p_{\bar{d}}) \Leftrightarrow (e_E, p_{\bar{c}}) < (e_E, p_{\bar{d}})$.

Proposition 35. Consider a standard quantum entity $S(\mathscr{E}_{sq}, \Sigma_{sq}, X_{sq}, \mathbb{O}_{sq}, \mathcal{M}_{sq})$. For $(e_E, p_{\bar{c}}), (e_F, p_{\bar{d}}) \in \mathscr{E}_{sq} \times \Sigma_{sq}$ we have

$$(e_E, p_{\bar{c}}) < (e_F, p_{\bar{d}}) \Leftrightarrow R(c) = c \quad \text{and} \quad T(c) = c \tag{185}$$

where

$$R = \sum_{x_{E_k} \in O(e_E\, p_{\bar{d}})} E_k, \qquad T = \sum_{E_k \in E \cap F} E_k \tag{186}$$

Proposition 36. Consider a standard quantum entity $S(\mathscr{E}_{sq}, \Sigma_{sq}, X_{sq}, \mathbb{O}_{sq}, \mathscr{M}_{sq})$. For $(e_E, p_{\bar{c}}), (e_F, p_{\bar{d}}) \in \mathscr{E}_{sq} \times \Sigma_{sq}$ we have

$$(e_E, p_{\bar{c}}) \perp (e_F, p_{\bar{d}}) \Leftrightarrow T(c) = c \quad \text{or} \quad T(d) = d \tag{187}$$

where

$$T = \sum_{E_k \notin E \cap F} E_k \tag{188}$$

The concept of eigenstates coincides with the traditional one.

15.2. The Eigenclosures

Let us construct the eigenclosures for the standard Hilbert space model. We can prove the following proposition:

Proposition 37. Consider a standard quantum entity $S(\mathscr{E}_{sq}, \Sigma_{sq}, X_{sq}, \mathbb{O}_{sq}, \mathscr{M}_{sq})$. For an experiment e_E with $E = \{E_1, \ldots, E_r\}$ and $A \subset O(e_E)$, we have

$$p_{\bar{c}} \in eig_{e_E}(A) \Leftrightarrow c \in R(A)(\mathscr{H}) \tag{189}$$

where

$$R(A) = \sum_{x_{E_k} \in A} E_k \tag{190}$$

Proof. $p_{\bar{c}} \in eig_{e_E}(A) \Leftrightarrow O(e_E, p_{\bar{c}}) \subset A \Leftrightarrow E_k(c) = 0$ for $x_{E_k} \notin A \Leftrightarrow R(A)(c) = c \Leftrightarrow c \in R(A)(\mathscr{H})$.

This proposition shows that the $eig_{e_E}(A)$ corresponds to the orthogonal projections or closed subspaces of the Hilbert space.

Proposition 38. Consider a standard quantum entity $S(\mathscr{E}_{sq}, \Sigma_{sq}, X_{sq}, \mathbb{O}_{sq}, \mathscr{M}_{sq})$. For an arbitrary R, orthogonal projection of $\mathscr{H}$, and the spectral set $E = \{R, 1 - R\}$ we have

$$c \in R(\mathscr{H}) \Leftrightarrow p_{\bar{c}} \in eig_{e_E}(\{R\}) \tag{191}$$

Proof. $c \in R(\mathscr{H}) \Leftrightarrow R(c) = c \Leftrightarrow (1 - R)(c) = 0 \Leftrightarrow O(e_E, p_{\bar{c}}) = \{R\} \Leftrightarrow p_{\bar{c}} \in eig_{e_E}(\{R\})$.

From these propositions it follows that the state eigenclosure system for the standard quantum mechanical model is isomorphic with the closure structure of the Hilbert space.

15.3. The Orthoclosures

Let us investigate the orthoclosure system of standard quantum mechanics and prove that the state orthoclosure system coincides completely with the state eigenclosure system.

Theorem 43. Consider a standard quantum entity $S(\mathscr{E}_{\mathrm{sq}}, \Sigma_{\mathrm{sq}}, X_{\mathrm{sq}}, \mathbb{O}_{\mathrm{sq}}, \mathscr{M}_{\mathrm{sq}})$. For $K \subset \Sigma_{\mathrm{sq}}$ and $L = \{c | p_{\bar{c}} \in K\}$ we have

$$K^{\perp} = \{p_{\bar{c}} | c \in L^{\perp}\} \tag{192}$$

$$cl_{orth}(K) = \{p_{\bar{c}} | c \in cl(L)\} \tag{193}$$

where cl is the closure operator in the Hilbert space. Suppose that L is a closed subspace of $\mathscr{H}$, and $F = \{p_{\bar{c}} | c \in L\}$; then we have $F \in \mathscr{F}_{orth}$. For the standard quantum mechanical mechanical model we have

$$\mathscr{F}_{eig} = \mathscr{F}_{orth} \tag{194}$$

Proof. We have $K^{\perp} = \{p_{\bar{c}} | p_{\bar{c}} \perp p_{\bar{d}}, p_{\bar{d}} \in K\} = \{p_{\bar{c}} | c \perp d, d \in L\} = \{p_{\bar{c}} | c \in L^{\perp}\}$. From this it follows that $cl_{orth}(K) = (K^{\perp})^{\perp} = \{p_{\bar{c}} | c \in (L^{\perp})^{\perp}\} = \{p_{\bar{c}} | c \in cl(L)\}$. Consider now L to be a closed subspace of the Hilbert space and $F = \{p_{\bar{c}} | c \in L\}$. Then $cl_{orth}(F) = \{p_{\bar{c}} | c \in cl(L)\} = \{p_{\bar{c}} | c \in L\} = F$, which shows that $F \in F_{orth}$.

So for the standard quantum mechanical formalism the eigenclosure system and the orthoclosure system coincide. As a consequence the eigenclosure system is orthocomplemented.

16. COMPLETED QUANTUM MECHANICS: A POSSIBLE SOLUTION OF THE SUBENTITY PROBLEM

For standard quantum mechanics a subentity is described by means of the tensor product procedure of the Hilbert spaces. Let us explain briefly how this procedure works. Let S and S' be described in complex Hilbert spaces $\mathscr{H}$ and $\mathscr{H}'$ such that $\mathscr{H}' = \mathscr{H} \otimes \mathscr{G}$, where $\mathscr{G}$ is another complex Hilbert space. In this situation 'standard quantum mechanics says that' the entity S' consists of two subentities, one described by the Hilbert space $\mathscr{H}$ (this is S) and one described by the Hilbert space $\mathscr{G}$ (let us call this entity T). We have studied this situation in detail in earlier work (Aerts and Daubechies, 1978; Aerts, 1984a), and will here only show how this scheme fits (and does not fit—and this will be the reason to 'change' standard quantum mechanics and formulate a new 'completed' quantum mechanics within Hilbert space) into the general description of a subentity that we have developed in this new approach.

Let us consider the entity $S(\mathscr{E}_{sq}, \Sigma_{sq}, X_{sq}, \mathbb{O}_{sq}, \mathscr{M}_{sq})$ described in the Hilbert space $\mathscr{H}$ and the entity $S'(\mathscr{E}'_{sq}, \Sigma'_{sq}, X'_{sq}, \mathbb{O}'_{sq}, \mathscr{M}'_{sq})$ described in the Hilbert space $\mathscr{H}'$ and suppose that S is a subentity of S'. Let us identify the connection functions m, n, l, and k. Let us first do this for the functions n and l, because we will see that we will hit upon a strange situation for the functions m and k. We have

$$n: \quad \mathscr{E}_{sq} \to \mathscr{E}'_{sq}, \qquad e_E \mapsto e'_{E'} = n(e_E) \tag{195}$$

$$E' = \{E_1 \otimes I_{\mathscr{G}}, E_2 \otimes I_{\mathscr{G}}, \ldots, E_k \otimes I_{\mathscr{G}}\} \tag{196}$$

$$l: \quad X_{sq} \to X'_{sq}, \qquad x_{E_k} \mapsto x'_{E'_{k'}} = l\,(x_{E_k}) \tag{197}$$

$$E'_{k'} = E_k \otimes I_{\mathscr{G}} \tag{198}$$

These two functions show that for the standard tensor product procedure of standard quantum mechanics we can make correspond with each experiment e_E on the subentity S a unique experiment $e'_{E'}$ on the big entity S', and also with each outcome x_{E_k} of the subentity S there corresponds a unique outcome $x'_{E'_{k'}}$ of the big entity S'.

The requirement that to each state $p'_{\bar{c}'}$ of the big entity S' there corresponds a unique state of the subentity S is not satisfied in this tensor product procedure within standard quantum mechanics. It is only met for some of the states of the big entity S', namely for the product states. Indeed if we consider a state $p'_{\bar{c}'}$, where $c' = c \otimes d$, the function m can be defined as follows: $m(p'_{\bar{c}'}) = p_{\bar{c}}$. But for a general state of S', and especially a nonproduct state, i.e., $p'_{\bar{c}'}$, where $c' = \Sigma_i\, c_i \otimes d_i$, this cannot be done.

Let us consider the natural correspondence between the probabilities of the subentity and the big entity, which defines the function k, and see that also here we have a correspondence only in the case that the big entity is in a product state. Consider a probability measure μ for the subentity, the big entity being in a product state $p_{\bar{c}'}$ with $c' = c \otimes d$. Hence we have $\mu\,(e_E, m(p'_{\bar{c}'}), E_k) = \langle c, E_k c\rangle$. The corresponding probability measure μ' for the big entity should be such that $\mu'(n(e_E), p_{\bar{c}'}, l(x_{E_k})) = \mu(e_E, m(p'_{\bar{c}'}), E_k)$. If we put $\mu'(n(e_E), p_{\bar{c}'}, l(x_{E_k})) = \langle \bar{c}', (E_k \otimes I_{\mathscr{G}})\bar{c}'\rangle$, then this is satisfied. Indeed we have $\langle \bar{c}', (E_k \otimes I_{\mathscr{G}})\bar{c}'\rangle = \langle c \otimes d, (E_k \otimes I_{\mathscr{G}}) c \otimes d\rangle = \langle c, E_k c\rangle\langle d, I_{\mathscr{G}} d\rangle = \langle c, E_k c\rangle\langle d, d\rangle = \langle c, E_k c\rangle$. So we can define $\mu' = k(\mu)$.

Our analysis means that the tensor product procedure of standard quantum mechanics cannot be used to describe subentities of the new approach. In Aerts (1984b) we show that some of the traditional axioms that lead to standard quantum mechanics are at the origin of this problem. More specifically these are the axioms of orthocomplementation, the covering law, and the axiom of atomicity (Aerts, 1984b). The problem of the description of compound entities and quantum axiomatics (which includes the problem of the description of subentities) has also been identified in other axiomatic approaches (Randall and Foulis, 1981; Pulmannova, 1983, 1984, 1985; Aerts and Valckenborgh, 1999) and possibilities to replace the failing axioms are under investigation (Aerts and Van Steirteghem, 1999).

16.1. The Subentity Problem in Standard Quantum Mechanics

We now consider a completely new possibility to solve this problem. If the 'solution' that we propose here is correct, this will automatically lead to the formulation of a new 'completed' quantum mechanics in Hilbert space. Let us explain how we came to this possible solution.

The main problem is that if the big entity is in a nonproduct state represented by a ray of the tensor product Hilbert space $\mathcal{H} \otimes \mathcal{G}$, the subentities are not in a state represented by a ray of one of the Hilbert spaces $\mathcal{H}$ or $\mathcal{G}$. This seems to indicate that the subentities 'are not in a state' even if the big entity 'is in a state.' This is of course very difficult to imagine. Indeed, if a piece of reality (the big entity) is in a certain state, then also a 'piece' of this 'piece' of reality (in this case the subentities) should be in a state. It is hard to conceive of a reality that would not satisfy such an elementary and fundamental property. Let us indicate the deep conceptual problem that we have just stated 'the subentity problem of standard quantum mechanics.'

Whereas this problem was known from the early days of quantum mechanics, it was concealed more or less by the confusion that often exists between pure states and mixtures. Let us explain this first. The reality of a quantum entity in standard quantum mechanics is represented by a pure state, namely a ray of the corresponding Hilbert space. So what we have called 'states' in this article are often called 'pure states.' Mixed states (what we also have called mixed states in this article—see Section 4) are represented in standard quantum mechanics by density matrices (positive self-adjoint operators with trace equal to 1). But although a mixed state is also called a state, it does not represent the reality of the entity under consideration, but a lack of knowledge about this reality. This means that if the entity is in a mixed state, it is actually in a pure state, and the mixed state just describes the lack of knowledge that we have about the pure state it is in. We have remarked that the deep conceptual problem that we indicate here was noticed already in the early days of quantum mechanics, but disguised by the existence of the two types of states, pure states and mixed states. Indeed in most books on quantum mechanics it is mentioned that for the description of subentities by means of the tensor product procedure the big entity can be in a pure state (and a nonproduct state is meant here) such that the subentities will be in mixed states and not in pure states (see, for example, Jauch, 1968, Section 11-8, and Cohen-Tannoudji, 1973, p. 306). The fact that the subentities, although not in a pure state, are at least in a mixed state seems at first sight to be some kind of a solution to the 'subentity problem in standard quantum mechanics'. A little further reflection, however, shows that it is not: indeed, if a subentity is in a mixed state, it should anyhow be in a pure state, and this mixed state should just describe our lack of knowledge about this pure state. So the 'subentity problem of standard quantum mechanics' is not solved at all. Probably because quantum mechanics is anyhow entailed with many paradoxes and mysteries, the deep problem of the subentity description was unconsciously just added to the list by the majority of physicists.

Way back in 1984 we already showed that in a more general approach we can define pure states for the subentities, but they will not be 'atoms' of the lattice of properties (Aerts, 1984b). Now it can easily be shown that within the general lattice approach (very similar to the approach that we have exposed in this paper in Section 6) standard quantum mechanics gives rise to an atomic property lattice, the rays of the Hilbert space representing the atoms of the lattice (see also Theorem 42 of this paper, which proves the 'state atomicity'). This means that the nonatomic pure states that we had identified in Aerts (1984b) cannot been represented within the standard quantum mechanical formalism. We must admit that the finding of the existence of nonatomic pure states in the 1984 paper, even from the point of view of generalized quantum formalisms, seemed also to us very far-reaching and difficult to interpret physically. Indeed intuitively it seems that only atomic states should represent pure states. We know now that this is a wrong intuition. But to explain why, we have to present first the other pieces of the puzzle.

A second piece of the puzzle appeared when in 1991 we built a model of a mechanistic classical laboratory situation violating the Bell inequalities with $\sqrt{2}$, exactly 'in the same way' as it is violated by the EPR experiments (Aerts, 1991). With this model we tried to show that the Bell inequalities can be violated in the macroscopic world with the same numerical value as the quantum violation. What is interesting for the problem of the description of subentities is that new 'pure' states were introduced in this model. We will see in a moment that the possibility of existence of these new states leads to a solution of the problem of the description of subentities within a Hilbert space setting, but different from standard quantum mechanics.

More pieces of the puzzle appeared steadily during the elaboration of the general formalism presented in the present paper. We started to work on this formalism during the first half of the 1980s, reformulating and elaborating some of the concepts over the years. Then it became clear that the new states introduced in Aerts (1991), although 'pure' states in the model, appear as nonatomic states in the general formalism. This made us understand that the first intuition that classified nonatomic states as not good candidates for pure states was a wrong intuition. Let us present now the total scheme of our solution.

16.2. The Quantum Machine: A Macroscopic Spin Model

We have introduced this example on earlier occasions (Aerts, 1986, 1991, 1995; Aerts and Durt, 1994) and will use it here to illustrate the solution of the subentity problem of standard quantum mechanics that we want to present and we will show how all the pieces of the puzzle fit together. The quantum machine is in fact a model for the spin of a spin-1/2 quantum entity. Let us present it in some detail such that this section is self-contained.

The entity S_{qm} that we consider is a point particle P that can move on the surface of a sphere denoted by *surf* with center 0 (the origin of a three-dimensional real space) and radius 1. The unit vector v giving the location of the particle on the surface of the sphere represents the state p_v of the particle (see Fig. 1a) when it is at the surface of the sphere. Hence the collection of all possible states of the entity S_{qm} that we consider is given by

$$\Sigma_{qm} = \{p_v | v \in surf\} \tag{199}$$

We define the following experiments. For each point $u \in surf$, we introduce the experiment e_u. We consider the diametrically opposite point $-u$, and install an elastic band of length 2 such that it is fixed with one of

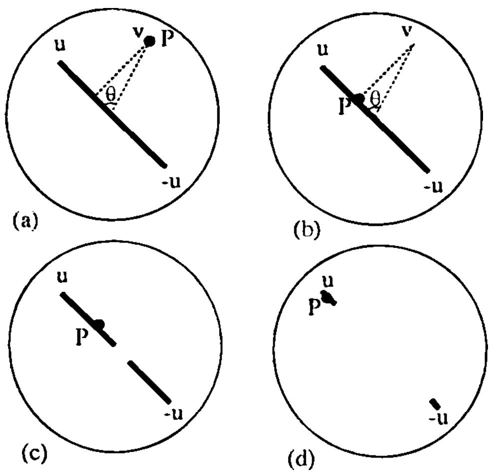

Fig. 1. A representation of the quantum machine. (a) The physical entity P is in state p_v at the point v, and an elastic corresponding to the experiment e_u is installed between the two diametrically opposed points u and $-u$. (b) The particle P falls orthogonally onto the elastic and sticks to it. (c) The elastic breaks and the particle P is pulled toward the point u, such that (d) it arrives at the point u, and the experiment e_u gets the outcome o_1^u.

its endpoints in u and the other endpoint in $-u$. Once the elastic is installed, the particle P falls from its original place v orthogonally onto the elastic and sticks on it (Fig. 1b). Then the elastic breaks and the particle P, attached to one of the two pieces of the elastic (Fig. 1c), moves to one of the two endpoints u or $-u$ (Fig. 1d). Depending on whether the particle P arrives at u (as in Fig. 1) or at $-u$, we give the outcome o_1^u or o_2^u to e_u. Hence for the quantum machine we have

$$\mathscr{E}_{qm} = \{e_u | u \in surf\} \tag{200}$$

If we consider the two unit vectors v, $u \in surf$, we can have the following possibilities. (1) If we have $v = u$, then $O(e_u, p_v) = \{o_1^u\}$; (2) if we have $v = -u$, then $O(e_u, p_v) = \{o_2^u\}$; (3) if we have $v \neq u$ and $v \neq -u$, then $O(e_u, p_v) = \{o_1^u, o_2^u\}$. This shows that

$$X_{qm} = \{o_1^u, o_2^u \mid u \in surf\} \tag{201}$$

The probabilities are easily calculated. The probability $\mu(e_u, p_v, o_1^u)$ that the particle P ends up at point u and hence experiment e_u gives outcome o_1^u is given by the length of the piece of elastic L_1 divided by the total length of the elastic. The probability $\mu(e_u, p_v, o_2^u)$ that the particle P ends up at point $-u$ and hence experiment e_u gives outcome o_2^u is given by the length of the piece of elastic L_2 divided by the total length of the elastic. This gives (Fig. 2)

$$\mu(e_u, p_v, o_1^u) = \frac{L_1}{2} = \frac{1}{2}(1 + \cos\theta) = \cos^2\frac{\theta}{2} \tag{202}$$

$$\mu(e_u, p_v, o_2^u) = \frac{L_2}{2} = \frac{1}{2}(1 - \cos\theta) = \sin^2\frac{\theta}{2} \tag{203}$$

These are exactly the standard quantum mechanical probabilities connected to the spin of a spin-1/2 quantum particle described in a two-dimensional complex Hilbert space.

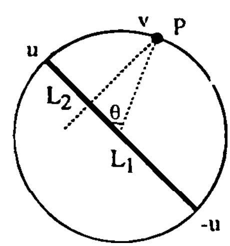

Fig. 2. A representation of the experimental process in the plane where it takes place. An elastic of length 2, corresponding to the experiment e_u, is installed between u and $-u$. The probability $\mu(e_u, p_v, o_1^u)$ that the particle P ends up at point u under the influence of the experiment e_u is given by the length of the piece of elastic L_1 divided by the total length of the elastic. The probability $\mu(e_u, p_v, o_2^u)$, that the particle P ends up at point $-u$ is given by the length of the piece of elastic L_2 divided by the total length of the elastic.

Let us present briefly also the quantum description. The state p_v is represented by $p_{\bar{c}^v}$, where

$$c^v = \left(\cos\frac{\theta}{2}e^{\frac{i\phi}{2}}, \sin\frac{\theta}{2}e^{\frac{-i\phi}{2}}\right) \tag{204}$$

and the experiment e_u is represented by e_{E^u}, where $E^u = \{E_1^u, E_2^u\}$ is the spectral family with spectral projections

$$E_1^u = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad E_2^u = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{205}$$

We remark that we have chosen the basis of the two-dimensional complex Hilbert space that describes our spin to coincide with the eigenvectors of e_u, hence $c^u = (1, 0)$ and $c^{-u} = (0, 1)$, but this does not endanger the generality of our description. Let us verify that the quantum mechanical calculation recovers the probabilities of our model. Indeed we have

$$\mu_q(e_{E^u}, p_{\bar{c}^v}, o_1) = \langle c^v, E_2^u\, c^v\rangle = \cos^2\frac{\theta}{2} = \mu(e_u, p_v, o_1^u) \tag{206}$$

$$\mu_q(e_{E^u}, p_{\bar{c}^v}, o_2) = \langle c^v, E_2^u\, c^v\rangle = \sin^2\frac{\theta}{2} = \mu(e_u, p_v, o_2^u)$$

This completes our model for the spin of a spin-1/2 quantum entity in standard quantum mechanics.

16.3. The New State Space: The Completed Quantum Machine

In the example that we proposed in Aerts (1991) we used two spin models like the one presented here and introduced new states on both models with the aim of presenting a situation that violates the Bell inequalities exactly as in the case of the singlet spin state of two coupled spin-1/2 particles. We indeed introduced a state for both spin models that corresponds to the point at the center of each sphere, and connecting these two states by a rigid rod, we could generate a violation of Bell's inequalities. Let us now introduce this state corresponding to the center 0 of the sphere explicitly and call it p_0. We clearly see that if we apply one of the experiments e_u to the point now in the state p_0, hence located at the center of the sphere, the probability corresponding to the respective outcomes is 1/2, and hence the set of possible outcomes is $\{o_1^u, o_2^u\}$ for any u. So we have

$$\mu(e_u, p_0, o_1^u) = 1/2, \qquad \mu(e_u, p_0, o_2^u) = 1/2\ \forall u \in surf \tag{207}$$

$$O(e_u, p_0) = \{o_1^u, o_2^u\} \qquad \forall u \in surf \tag{208}$$

If we consider the general definition of the 'state implication' introduced in Definition 2, then we can see that

$$p_v < p_0, \qquad p_0 \not< p_v, \qquad \forall v \in surf \tag{209}$$

which shows that p_0 is 'not an atom' of the preordered set of states. This means that we have 'identified' a possible 'nonmixture' state (meaning by 'nonmixture' that it really represents the reality of the entity and not a lack of knowledge about this reality) that is not an atom of the preordered set of states. Is this a candidate for the 'nonmixed' states that we identified in Aerts (1984b) and that were nonatoms? We will see that it is. Let us explicitly define all the new states that we want to introduce in our example. Since it will no longer be the same example, we will call this new quantum machine the 'completed' quantum machine.

The entity S_{cqm} (completed quantum machine) that we consider is again a point particle P that can move inside and on the surface of a sphere denoted by $ball = \{w | \, \|w\| \leq 1\}$ with center 0 (the origin of a three-dimensional real space) and radius 1. The vector w giving the location of the particle inside the sphere represents the state p_w of the particle (see Fig. 3). The experiments that we consider for this completed quantum machine are the same as the one we considered for the quantum machine. This means that the set of outcomes and the set of experiments are given by

$$\Sigma_{cqm} = \{p_w | w \in ball\}, \qquad \mathscr{E}_{cqm} = \{e_u | u \in surf\} \tag{210}$$

Before we calculate the probabilities for the completed quantum entity we remark the following. Because the sphere is a convex set, each vector $w \in ball$ can be written as a convex linear combination of two vectors v and $-v$ on the surface of the sphere (see Fig. 3). More concretely this means that we can write (referring to the w, v, and $-v$ in Fig. 3)

$$w = a \cdot v - b \cdot v, \quad a, b \leq 1, \quad a + b = 1 \tag{211}$$

Hence, if we introduce these convex combination coefficients a, b we have $w = (a - b) \cdot v$. Let us calculate now the transition probabilities for a completed quantum machine entity being in a general state p_w with $w \in ball$

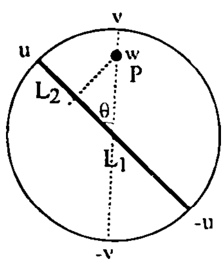

Fig. 3. A representation of the experimental process in the case of the 'completed' quantum machine. An elastic of length 2, corresponding to the experiment e_u, is installed between u and $-u$. The probability $\mu(e_u, p_w, o_1^u)$ that the particle P ends up at point u under influence of the experiment e_u is given by the length of the piece of elastic L_1 divided by the total length of the elastic. The probability $\mu(e_u, p_w, o_2^u)$ that the particle P ends up at point $-u$ is given by the length of the piece of elastic l_{-2} divided by the total length of the elastic.

and hence $\|w\| \leq 1$ (see Fig. 3). Again the probability $\mu(e_u, p_w, o_1^u)$ that the particle P ends up at point u and hence experiment e_u gives outcome o_1^u is given by the length of the piece of elastic L_1 divided by the total length of the elastic. The probability that $-u$ and hence experiment e_u gives outcome o_2^u is given by the length of the piece of elastic L_2 divided by the total length of the elastic. This means that we have

$$\mu(e_u, p_w, o_1^u) = \frac{L_1}{2} = \frac{1}{2}(1 + (a - b)\cos\theta) = a\cos^2\frac{\theta}{2} + b\sin^2\frac{\theta}{2} \quad (212)$$

$$\mu(e_u, p_w, o_2^u) = \frac{L_2}{2} = \frac{1}{2}(1 - (a - b)\cos\theta) = a\sin^2\frac{\theta}{2} + b\cos^2\frac{\theta}{2} \quad (213)$$

These are new probabilities that will never be obtained if we limit the set of states to the rays of the two-dimensional complex Hilbert space as is the case for the (noncompleted) quantum machine. The question is now the following: can we find another mathematical entity, connected in some way or another to the Hilbert space, that would allow us, with a new quantum rule for calculating probabilities, to find these probabilities? The answer is yes, but now we have to proceed very carefully not to get into too much confusion. We will show that these new 'pure' states of the interior of the sphere can be represented by using density matrices, the same matrices that are used within the standard quantum formalism to represent mixed states. And the standard quantum mechanical formula that is used to calculate the probabilities connected to mixed states, represented by density matrices, can also be used to calculate the probabilities that we have identified here. But of course the meaning will be different: in our case this standard formula will represent a transition probability from one pure state to another and not the probability connected to the change of a mixed state. Let us show all this explicitly and to do this construct the density matrices in question.

The well-known quantum formula for the calculation of the probabilities for an outcome x_{E_k} if an experiment e_E is performed, where $E = \{E_1, \ldots, E_k, \ldots, E_n\}$, is the spectral decomposition corresponding to the experiment, and where the quantum entity is in a mixed state p represented by the density matrix W, is the following:

$$\mu(e_E, p, E_k) = tr(W \cdot E_k) \quad (214)$$

where tr is the trace of the matrix.

A standard quantum mechanical calculation shows that the density matrix representing the ray state

$$c_v = \left(\cos\frac{\theta}{2}e^{i\phi/2}, \sin\frac{\theta}{2}e^{-i\phi/2}\right)$$

[see (204)] is given by

$$W(v) = \begin{pmatrix} \cos^2 \dfrac{\theta}{2} & \sin \dfrac{\theta}{2} \cos \dfrac{\theta}{2} e^{-i\phi} \\ \sin \dfrac{\theta}{2} \cos \dfrac{\theta}{2} e^{i\phi} & \sin^2 \dfrac{\theta}{2} \end{pmatrix} \tag{215}$$

and the density matrix representing the diametrically opposed ray state c_{-v} is given by

$$W(-v) = \begin{pmatrix} \sin^2 \dfrac{\theta}{2} & -\sin \dfrac{\theta}{2} \cos \dfrac{\theta}{2} e^{-i\phi} \\ -\sin \dfrac{\theta}{2} \cos \dfrac{\theta}{2} e^{i\phi} & \cos^2 \dfrac{\theta}{2} \end{pmatrix} \tag{216}$$

We will show now that the convex linear combination of these two density matrices with convex weights a and b represents the state p_w if we use the standard quantum mechanical formula [formula (214)] to calculate the transition probabilities. If, for $w = av + b(-v)$, we put

$$W(w) = aW(v) + bW(-v) \tag{217}$$

we have

$$W(w) = \begin{pmatrix} a \cos^2 \dfrac{\theta}{2} + b \sin^2 \dfrac{\theta}{2} & (a - b) \sin \dfrac{\theta}{2} \cos \dfrac{\theta}{2} e^{-i\phi} \\ (a - b) \sin \dfrac{\theta}{2} \cos \dfrac{\theta}{2} e^{i\phi} & a \sin^2 \dfrac{\theta}{2} + b \cos^2 \dfrac{\theta}{2} \end{pmatrix} \tag{218}$$

and it is easy to calculate now the transition probabilities using formula (214). We have

$$W(w) \cdot E_1 = \begin{pmatrix} a \cos^2 \dfrac{\theta}{2} + b \sin^2 \dfrac{\theta}{2} & 0 \\ (a - b) \sin \dfrac{\theta}{2} \cos \dfrac{\theta}{2} e^{i\phi} & 0 \end{pmatrix} \tag{219}$$

and hence, comparing with formula (212), we find

$$tr(W(w) \cdot E_1) = a \cos^2 \frac{\theta}{2} + b \sin^2 \frac{\theta}{2} = \mu(e_u, p_w, o_1^u) \tag{220}$$

In an analogous way we find that

$$tr(W(w) \cdot E_2) = a \sin^2 \frac{\theta}{2} + b \cos^2 \frac{\theta}{2} = \mu(e_u, p_w, o_2^u) \tag{221}$$

So we have shown that we can represent each one of the new states p_w by the density matrix $W(w)$ if we use formula (214) for the calculation of the transition probabilities.

Let us also prove that each density operator represents one of the new states p_w. We can show this easily by using the general properties of density matrices. Since a density operator is a self-adjoint operator, we can find an orthonormal base of the two-dimensional Hilbert space where it is diagonal. Since it is a positive operator with trace equal to 1 it will have two real numbers a, b such that $0 \leq a \leq 1$ and $0 \leq b \leq 1$ and $a + b = 1$ on its diagonal. Suppose that v and $-v$ are the diametrically opposed points of the sphere representing the base vectors. Then the density operator represents the state corresponding to the point $(a - b)v$.

Although we have done all the calculations here only explicitly for the case of a two-dimensional complex Hilbert space representing the spin of a spin-1/2 quantum entity, it can be shown easily that this procedure is generally valid for an arbitrary quantum entity with an arbitrary dimensional Hilbert space. The new nonproduct (hence pure) states that we need to introduce to solve the 'subentity problem' of standard quantum mechanics can be represented in a similar way by density operators. We show in much more detail the new aspect of this new approach to Hilbert space quantum mechanics in a forthcoming paper (Aerts, 1999).

We have not yet properly defined for the general case what is a density operator. Let us do this now such that we can prove that the step that we want to propose, namely interpreting the density operators as 'also' representing 'pure' states within a new 'completed' Hilbert space formalism, solves our original 'subentity problem.'

16.4. Completed Quantum Mechanics

A density operator W in the case of a general complex Hilbert space $\mathcal{H}$ is a positive self-adjoint operator with trace equal to 1. Only if $W^2 = W$ does it represent a projection operator on a ray of the Hilbert space and hence a 'ray state.' If $W^2 \neq W$, the density operator represents one of the new states that is not a ray state, but is still a pure state. The same density operator of course also still represents a mixed state as in the standard quantum mechanics. We remark that even in the standard quantum mechanics several distinct mixed states are represented by the same density operator, such that the 'double' representation that we introduce for this mathematical object does not lead to additional conceptual problems. We just have to be aware of for which type of state we use the specific representation of a specific density operator.

To resolve the confusion with the different types of states and their representations we will now introduce some new concepts.

Definition 51. Consider a separable complex Hilbert space $\mathcal{H}$. We introduce the set of density operators $\mathcal{W}(\mathcal{H})$. A density operator is a positive self-adjoint operator with trace equal to 1. We will denote density operators by $W, V, \ldots$.

The set of all density operators $\mathcal{W}(\mathcal{H})$ is a convex set, the subspace of the vector space of all bounded operators. This means that if we consider a set $(W_i)_i$ of density operators and a set $(a_i)_i$ of real numbers such that $\Sigma_i\, a_i = 1$, then $\Sigma_i\, a_i W_i$ is also a density operator. It can be shown that for $W \in \mathcal{W}(\mathcal{H})$ we have $W^2 = W$ iff W is an orthogonal projection on a one-dimensional subspace of $\mathcal{H}$. The density operators that equal their product are the extremal points of the convex set $\mathcal{W}(\mathcal{H})$ and they represent the 'ray' states. This also means that every density operator can be written as the convex sum of such ray state density operators. We now have all the necessary material to present a formal definition of a completed quantum entity.

Definition 52. Consider a probabilistic entity $S(\mathcal{E}_{\text{cq}}, \Sigma_{\text{cq}}, X_{\text{cq}}, \mathcal{O}_{\text{cq}}, \mathcal{M}_{\text{cq}})$ and a separable complex Hilbert space $\mathcal{H}$, with a set of density operators $\mathcal{W}(\mathcal{H})$, a set of orthogonal projections $\mathcal{P}(\mathcal{H})$, and a set of spectral families $\mathcal{S}(\mathcal{H})$. We say that the entity is a 'completed quantum entity' iff we have

$$\begin{aligned}
\mathcal{E}_{\text{cq}} &= \{e_E | E \in \mathcal{S}(\mathcal{H})\} \\
\Sigma_{\text{cq}} &= \{p_w | W \in \mathcal{W}(\mathcal{H})\} \\
X_{\text{cq}} &= \{x_{E_k} | E_k \in \mathcal{P}(\mathcal{H})\} \qquad (222) \\
\mathcal{O}_{\text{cq}} &= \{O(e_E, p_w) | E \in \mathcal{S}(\mathcal{H}),\ W \in \mathcal{W}(\mathcal{H})\} \\
\mathcal{M}_{\text{cq}} &= \{\mu | \mu\colon \mathcal{E}_{\text{cq}} \times \Sigma_{\text{cq}} \times X_{\text{cq}} \rightarrow [0, 1] \text{ is a generalized probability}\}
\end{aligned}$$

such that

$$\begin{aligned}
O(e_E, p_W) &= \{x_{E_k} | E_k \in \mathcal{P}(\mathcal{H}),\ E_k \in E,\ tr(WE_k) \neq 0\} \\
\mu(e_E, p_w, E_k) &= tr(WE_k) \qquad \text{if} \quad E_\kappa \in E \qquad (223) \\
\mu(e_E, p_w, E_k) &= 0 \qquad \text{if} \quad E_k \notin E
\end{aligned}$$

For a completed quantum entity we can solve the problem of the description of the subentity. Let us consider again the situation of a completed quantum entity $S(\mathscr{E}_{cq}, \Sigma_{cq}, X_{cq}, \mathscr{O}_{cq}, \mathscr{M}_{cq})$, described in a Hilbert space $\mathscr{H}$ that is a subentity of a completed quantum entity $S'(\mathscr{E}'_{cq}, \Sigma'_{cq}, X'_{cq}, \mathscr{O}'_{cq}, \mathscr{M}'_{cq})$ described in a Hilbert space $\mathscr{H}'$.

The functions n and l are defined as in the case of standard quantum mechanics, namely

$$n\colon \quad \mathscr{E}_{cq} \to \mathscr{E}_{cq} \qquad e_E \mapsto e'_{E'} = n(e_E) \tag{224}$$

$$E' = \{E_1 \otimes I_{\mathscr{G}}, E_2 \otimes I_{\mathscr{G}}, \ldots, E_k \otimes I_{\mathscr{G}}\} \tag{225}$$

$$l\colon \quad X_{cq} \to X'_{cq}, \qquad x_{E_k} \mapsto x'_{E'k'} = l(x_{E_k}) \tag{226}$$

$$E'_{k'} = E_k \otimes I_{\mathscr{G}} \tag{227}$$

Let us now consider a state $p'_{W'}$, of the big entity S'. Let us show that there is one unique state $m(p'_{W'}) = p_W$ of the entity S such that $tr(W' E'_{k'}) = tr(W E_k)$ and hence $\mu(e_E, m(p'_{W'}), E_k) = k(\mu)(n(e_E), p'_{W'}, i(E_k))$.

Proposition 39. Let us suppose that we have three Hilbert spaces $\mathscr{H}$, $\mathscr{G}$, and $\mathscr{H}'$ such that $\mathscr{H}' = \mathscr{H} \otimes \mathscr{G}$. For a density operator $W' \in \mathscr{W}'(\mathscr{H}')$ there exists a unique density operator $W \in \mathscr{W}(\mathscr{H})$ such that for an arbitrary $E_\kappa \in \mathscr{P}(\mathscr{H})$ we have $tr(W' E_k \otimes I_{\mathscr{G}}) = tr(W E_k)$. We will denote $W = \hat{m}(W')$.

Proof. We first prove that W is unique if it exists. Suppose that we would have two density operators $W, V \in \mathscr{W}(\mathscr{H})$ such that $tr(W' E_k \otimes I_{\mathscr{G}}) = tr(W E_k) = tr(V E_k)\ \forall E_k \in \mathscr{P}(\mathscr{H})$. If we consider especially the projection operator $E_{\bar{c}}$ on an arbitrary ray $\bar{c}$ of the Hilbert space $\mathscr{H}$, then we have $tr(W E_{\bar{c}}) = \langle c, Wc\rangle = \langle c, Vc\rangle = tr(V E_{\bar{c}})$. This shows that $\langle c, Wc\rangle = \langle c, Vc\rangle$ $\forall c \in \mathscr{H}$ and as a consequence $W = V$.

Suppose that W is a solution for an arbitrary W'. We know that W' can be written as the convex sum $\Sigma_{c'} a(c') W_{c'}$ of density operators $W'_{c'}$ corresponding to projections on the rays $\bar{c}'$, and hence with $\Sigma_{c'} a(c') = 1$. Due to the linearity of the trace we have $tr(W' E_k \otimes I_{\mathscr{G}}) = \Sigma_{c'}\, a(c')\, tr(W'_{c'} E_k)$, which shows that if we construct the density operator W for the case where $W' = W'_{c'}$ is a density operator corresponding to a ray $\bar{c}'$; we have a solution for the general situation.

This means that we have only to construct a solution for the case of a density operator $W'_{c'}$ corresponding to a ray $\bar{c}'$ of the big entity S'. Let us first show that W has trace equal to 1. Suppose that we consider an orthonormal base $(c_i)_i$ of $\mathscr{H}$ and an orthonormal base $(d_j)_j$ of $\mathscr{G}$; then $(c_i \otimes d_j)_{ij}$ is an orthonormal base of $\mathscr{H}'$. This means that we can write $c' = \Sigma_{ij}\, a_{ij} c_i \otimes d_j$. We have:

$$\begin{aligned} 1 = \langle c', c' \rangle &= \sum_{ijkl} a_{ij} a_{kl} \langle c_i \otimes d_j, c_k \otimes d_l \rangle \\ &= \sum_{ijkl} a_{ij} a^*_{kl} \langle c_i, c_k \rangle \langle d_j, d_l \rangle \\ &= \sum_{ijkl} a_{ij} a^*_{kl} \delta_{ik} \delta_{jl} = \sum_{ij} \|a\|^2_{ij} \end{aligned} \tag{228}$$

Let us now use the correspondence of the probabilities as required by the subentity relation. We have $tr(W\, E_k) = tr(W' E_k \otimes I_{\mathscr{G}})$ for all $E_k \in \mathscr{P}(\mathscr{H})$. Take especially E_k to be the projector on c_m, and let us denote this projector by E_m. Then we have

$$\begin{aligned} \langle c_m, W\, c_m \rangle = tr(W\, E_m) &= tr(W' E_m \otimes I_{\mathscr{G}}) = \langle c', E_m \otimes I_{\mathscr{G}} c' \rangle \\ &= \langle \sum_{ij} a_{ij} c_i \otimes d_j, E_m \otimes I_{\mathscr{G}} \sum_{kl} a_{kl}\, c_k \otimes d_l \rangle \\ &= \langle \sum_{ij} a_{ij} c_i \otimes d_j, \sum_{kl} a_{kl} (E_m c_k) \otimes d_l \rangle \\ &= \langle \sum_{ij} a_{ij} c_i \otimes d_j, \sum_{l} a_{ml} c_m \otimes d_l \rangle \\ &= \sum_{ijl} a_{ij} a^*_{ml} \delta_{im} \delta_{jl} \\ &= \sum_{j} \|a_{mj}\|^2 \end{aligned} \tag{229}$$

This shows that

$$tr(W) = \sum_{m} \langle c_m, W\, c_m \rangle = \sum_{mj} \|a_{mj}\|^2 = 1 \tag{230}$$

We can easily calculate, using (229), the matrix elements of W in a base where W is diagonal (this always exists since W is a self-adjoint operator).

The result of proposition 39 makes it possible for us to define unambiguously the functions m and k. Indeed;

$$m\colon\quad \Sigma'_{\mathrm{cq}} \to \Sigma_{\mathrm{cq}}, \qquad p_{W'} \mapsto p_W = m(p_{W'}) \tag{231}$$

$$W = \hat{m}(W') \tag{232}$$

$$k\colon\quad \mathscr{M}_{\mathrm{cq}} \to \mathscr{M}'_{\mathrm{cq}}, \qquad \mu \mapsto k(\mu) \tag{233}$$

$$\begin{aligned} tr(W\, E_k) &= tr(\hat{m}(W') E_k) = \mu(e_E, m(p_{W'}), x_{Ek}) \\ &= k(\mu)(n(e_E), p_{W'}, l(x_{E_k})) = tr(W' E_k \otimes I_{\mathscr{G}}) \quad \text{for} \quad E_k \in E \end{aligned} \tag{234}$$

17. CONCLUSION

We announced in the Introduction that we would elaborate as essential components of a general operational and realistic formalism the structures of the states, the experiments, the outcomes, the probabilities, and the symmetries. We have treated the structures of the states, experiments, and outcomes in some detail and point out now the aspects that are still missing and will be presented in forthcoming work. If we think of Piron's representation theorem (Piron, 1976), which is formulated within the category of state property systems (Aerts, *et al.*, 1999), it takes (1) completeness, (2) atomicity, (3) orthocomplementation, (4) weak modularity, and (5) the covering law to arrive at a structure that is isomorphic with a generalized Hilbert space. For an updated version of the axioms necessary for this representation theorem, also incorporating the resent result of Solèr, we refer to Aerts and Van Steirteghem (1999). We have treated the completeness and the atomicity in the formalism presented here. We have shown that the completeness of the whole set of properties can only be derived for the case of distinguishable experiment entities and we have proven that the atomicity is equivalent to the T_1 separation property for the eigenclosure structure. We have introduced the orthoclosure structure and this closure structure gives rise in a natural way to an orthocomplementation. This is the reason that it is possible to introduce the orthocomplementation by postulating that the eigenclosure structure has to coincide with the orthoclosure structure, as proposed in Aerts (1994). We have not made this step in this article because we want to study the problem of the introduction of an orthocomplementation in a more detailed way in forthcoming work. We mention that because we made the choice to treat the states and the properties of an entity as independent concepts, which was not the case in the earlier approaches, we identified a new axiom, which we have called 'state determination' (see also Aerts, 1994). We have not touched on weak modularity and the covering law: this will be done in future work. We have also only briefly introduced the concept of probability and left the elaboration of it for future investigation. We have not spoken at all of the symmetries and want to mention briefly how we will analyze this aspect. Considering the group of automorphisms of our basic mathematical structure, we want to introduce the symmetries as group representations of the different physical groups that are connected to the different symmetries.

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REFERENCES

Aerts, D. (1981). The one and the many, Doctoral Thesis, Free University of Brussels, Brussels.

Aerts, D. (1982). Description of many physical entities without the paradoxes encountered in quantum mechanics, *Found. Phys.* **12**, 1131.

Aerts, D. (1983a). Classical theories and non classical theories as a special case of a more general theory, *J. Math. Phys.* **24**, 2441.

Aerts, D. (1983b). The description of one and many physical systems, in *Foundations of Quantum Mechanics*, C. Gruber, ed., A.V.C.P., Lausanne, p. 63.

Aerts, D. (1984a). Construction of a structure which makes it possible to describe the joint system of a classical and a quantum system, *Rep. Math. Phys.* **20**, 421.

Aerts, D. (1984b). Construction of the tensor product for lattices of properties of physical entities, *J. Math. Phys.* **25**, 1434.

Aerts, D. (1986). A possible explanation for the probabilities of quantum mechanics, *J. Math. Phys.* **27**, 202.

Aerts, D. (1991). A mechanistic classical laboratory situation violating the Bell inequalities with $\sqrt{2}$, exactly 'in the same way' as its violations by the EPR experiments, *Helv. Phys. Acta* **64**, 1.

Aerts, D. (1994). Quantum structures, separated physical entities and probability, *Found. Phys.* **24**, 1227.

Aerts, D. (1995). Quantum structures: An attempt to explain their appearance in nature, *Int. J. Theor. Phys.* **34**, 1165.

Aerts, D. (1999). A possible solution of the sub entity problem of standard quantum mechanics leading to a new type of Hilbert space quantum mechanics, Preprint FUND-CLEA, Free University of Brussels.

Aerts, D. and Daubechies, I. (1978). Physical justification for using the tensor product to describe two quantum systems as one joint system, *Helv. Phys. Acta* **51**, 661.

Aerts, D., and Durt, T. (1994). Quantum, classical and intermediate, an illustrative example, *Found. Phys.* **24**, 1353.

Aerts, D., and Valckenborgh, F. (1999). Lattice extensions and the description of compound entities, FUND, Brussels Free University, Preprint.

Aerts, D., and Van Steirteghem, B., (1999). Quantum axiomatics and a theorem of M.P. Solèr, *International Journal of Theoretical Physics*, submitted.

Aerts, D., Coecke, B., Durt, T., and Valckenborgh, F. (1997a). Quantum, classical and intermediate; a model on the Poincaré sphere, *Tatra Mountains Math. Publ.* **10**, 225.

Aerts, D., Coecke, B., Durt, T., and Valckenborgh, F. (1997b). Quantum, classical and intermediate; the vanishing vector space structure, *Tatra Mountains Math. Publ.* **10**, 241.

Aerts, D., Colebunders, E., Van der Voorde, A., and Van Steirteghem, B. (1999). State property systems and closure spaces: A study of categorical equivalence, *Int. J. Theor. Phys.*, this issue.

Birkhoff, G., (1973). *Lattice Theory, Amer. Math. Soc.,* Colloq. Publ. Vol. XXV, Providence.

Birkhoff, G., and Von Neumann, J. (1936). The logic of quantum mechanics, *Ann. Math.* **37**, 823.

Beltrametti, E., and Cassinelli, G. (1981). The logic of quantum mechanics, Reading, Mass. Addison-Wesley.

Cohen-Tannoudji, C., Diu, B., and Laloë, F. (1973). *Mécanique Quantique*, Vol. I, Hermann, Paris.

Foulis, D., and Randall, C. (1981). What are quantum logics and what ought they to be? in *Current Issues in Quantum Logic*, E. Beltrametti and B. van Fraassen, eds., Plenum Press, New York, p. 35.

Foulis, D., Piron, C., and Randall, C. (1983). Realism, operationalism, and quantum mechanics, *Found. Phys.* **13**, 813.

Holland, S. (1995). *Bull. of the Amer. Math. Soc.,* **32**, 205.

Jauch, J. (1968). *Foundations of Quantum Mechanics*, Addison-Wesley, Reading, Massachusetts.

Keller, H. A.(1980), *Math. Z.,* **172**, 41.

Ludwig, G. (1983), Foundations of Quantum Mechanics, New York and Berlin, Springer-Verlag.

Ludwig, G. (1985), An Axiomatic Basis for Quantum Mechanics, New York and Berlin, Springer-Verlag.

Mackey, G. W. (1963). *Mathematical Foundations of Quantum Mechanics*, Benjamin, Reading, Massachusetts.

Piron, C. (1964). Axiomatique quantique, *Helv. Phys. Acta* **37**, 439.

Piron, C. (1976). *Foundations of Quantum Physics*, Benjamin, Reading, Massachusetts.

Piron, C. (1989). Recent developments in quantum mechanics, *Helv. Phys. Acta* **62**, 82.

Piron, C. (1990). *Mècanique Quantique: Bases et Applications*, Press Polytechnique de Lausanne, Lausanne, Switzerland.

Pulmannova, S. (1983). Coupling of quantum logics, *Int. J. Theor. Phys.* **22**, 837.

Pulmannova, S. (1984). On the product of quantum logics, *Suppl. Circ. Mat. Palermo Ser. II*, 231.

Pulmannova, S. (1985). Tensor products of quantum logics, *J. Math. Phys.* **26**, 1.

Pulmannova, S. (1996). *Int. J. Theor. Phys.,* **35**, 2309.

Randall, C., and Foulis, D. (1976). A mathematical setting for inductive reasoning, in *Foundations of Probability Theory, Statistical Inference, and Statistical Theories of Science III*, C. Hooker, ed., Reidel, Dordrecht, p. 169.

Randall, C., and Foulis, D. (1978). The operational approach to quantum mechanics, in *Physical Theories as Logico-Operational Structures*, C. A. Hooker, ed., Reidel, Dordrecht, p. 167.

Randall, C., and Foulis, D. (1981). Operational statistics and tensor products, in *Interpretations and Foundations of Quantum Theory*, H. Neumann, ed., B. I. Wissenschaftsverslag, Bibliographisches Institut, Mannheim, p. 21.

Randall, C., and Foulis, D. (1983). Properties and operational propositions in quantum mechanics, *Found. Phys.* **13**, 835.

Solèr, M. P. (1995), *Comm. Alg.,* **23**, 219.

Von Neumann, J. (1932). *Mathematische Grundlagen der Quanten-Mechanik*, Springer-Verlag, Berlin.

Varadarajan, V. (1968). *Geometry of Quantum Theory*, von Nostrand, Princeton, New Jersey.

Wilbur, (1977). *Trans. Amer. Math. Soc.,* **233**, 265.

Zierler, N. (1961). Axioms for non-relativistic quantum mechanics, *Pac. J. Math.* **11**, 1151.